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The solvable Lie group $N_{6,28}$: an example of an almost $C_0(\mathcal{K})$ -C*-algebra

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Abstract

Motivated by the description of the C*-algebra of the affine automorphism group $N_{6,28}$ of the Siegel upper half-plane of degree 2 as an algebra of operator fields defined over the unitary dual $\widehat{N_{6,28}}$ of the group, we introduce a family of C*-algebras, which we call almost $C_0(\mathcal{K})$, and we show that the C*-algebra of the group $N_{6,28}$ belongs to this class.

1 Introduction

In order to analyze a C*-algebra \mathcal{A} , one can use the Fourier transform \mathcal{F} , which allows us to decompose \mathcal{A} over its unitary spectrum $\widehat{\mathcal{A}}$. To be able to define this transform, consider the algebra $l^\infty(\widehat{\mathcal{A}})$ of all bounded operator fields over $\widehat{\mathcal{A}}$ defined by

$$l^\infty(\widehat{\mathcal{A}}) := \{A = (A(\pi) \in \mathcal{B}(\mathcal{H}_\pi))_{\pi \in \widehat{\mathcal{A}}}; \|A\|_\infty := \sup_{\pi \in \widehat{\mathcal{A}}} \|A(\pi)\|_{\text{op}} < \infty\},$$

where \mathcal{H}_π is the Hilbert space of π . The space $l^\infty(\widehat{\mathcal{A}})$ is a (huge) C*-algebra itself. The Fourier transform \mathcal{F} defined by

$$\mathcal{F}(a) = \hat{a} := (\pi(a))_{\pi \in \widehat{\mathcal{A}}} \quad \text{for } a \in \mathcal{A}$$

is then an injective, hence isometric, homomorphism from \mathcal{A} into $l^\infty(\widehat{\mathcal{A}})$. The problem is now to recognize the elements of $\mathcal{F}(\mathcal{A})$ inside this big algebra $l^\infty(\widehat{\mathcal{A}})$.

Recall that a Lie group G is called *exponential* if it is a connected, simply connected solvable Lie group for which the exponential mapping $\exp : \mathfrak{g} \rightarrow G$ from the Lie algebra \mathfrak{g} to its Lie group G is a diffeomorphism. The Kirillov-Bernat-Vergne-Pukanszky-Ludwig-Leptin theory shows that there is a canonical homeomorphism $K : \mathfrak{g}^*/G \rightarrow \widehat{G}$ from the space of coadjoint orbits of G in the linear dual space \mathfrak{g}^* onto the unitary dual space \widehat{G} of G (see [Lep-Lud] for details and references). Then the unitary spectrum $\widehat{C^*(G)}$ of the C*-algebra $C^*(G)$ of the locally compact group G can be identified with the unitary dual \widehat{G} of G .

Since connected Lie groups are second countable, the algebra $C^*(G)$ and its dual space \widehat{G} are separable topological spaces. This allows us to work in \widehat{G} with sequences instead of nets. Moreover, if G is amenable, then the left regular representation $(\lambda, L^2(G))$ of $C^*(G)$, defined by $\lambda(F)\xi := F * \xi$ for $F \in L^1(G)$ and $\xi \in L^2(G)$, is injective. So we can also identify $C^*(G)$ with an algebra of convolution operators on the Hilbert space $L^2(G)$.

The method of describing group C^* -algebras as algebras of operator fields defined on the dual spaces of the groups has been studied in [Fe] and [Lee]. In [Lin-Lud1], the C^* -algebra of $ax + b$ -like groups has been characterized as the algebra of operator fields, while the C^* -algebra of the Heisenberg groups and of the threadlike groups are described in [Lu-Tu]. In both cases, the topologies of the orbit spaces \mathfrak{g}^*/G of the groups are well understood (see for instance [Ar-Lu-Sc]). The isomorphism problem for C^* -algebras of $ax + b$ -like groups was solved in [Lin-Lud2].

In this paper, we consider the Lie group G_6 , whose Lie algebra is the 6-dimensional normal j -algebra \mathfrak{g}_6 (which has been classified as $N_{6,28}$ by P. Turkowski in [Tu]); a prototype of this class of Lie algebras can be found, for example, in [In]. As a transformation group, our G_6 is the Iwasawa AN-group of $Sp(2, \mathbb{R})$. For a geometric background of this class of Lie groups as affine automorphisms of Siegel domains, we refer the reader, for example, to the textbook [Ka]. Thanks to the orbit structure of exponential groups, we can write down the dual space \widehat{G}_6 of our group G_6 and determine its topology. The main difference here from the previous cases, the $ax + b$ -like groups, is that there are now coadjoint orbits of dimension 0, 2, 4 and 6, respectively, which decompose the orbit space into the union of a sequence $(S_i)_{i=0}^6$ of seven increasing closed subsets. On each of the sets $\Gamma_i = S_i \setminus S_{i-1}$, $i = 1, \dots, 6$, the orbit space topology is Hausdorff and the main difficulty is to understand for a given $a \in C^*(G)$ the behaviour of the operators $\hat{a}(\gamma)$ for $\gamma \in \Gamma_i$, when γ approaches elements in S_{i-1} . For each of these sets Γ_i , we obtain different conditions for the C^* -algebra, conditions which we shall call the *continuity*, the *infinity* and the *boundary* conditions.

Our example motivates the introduction of a special class of C^* -algebras which we call *almost* $C_0(\mathcal{K})$, where \mathcal{K} is the algebra of all compact operators on a certain Hilbert space (Section 2). In Section 3, we describe the exponential Lie group G_6 , its Lie algebra \mathfrak{g}_6 and its coadjoint orbits in \mathfrak{g}_6^* . Each of them needs a special treatment which we describe in the following sections. In Section 4, we present the topology of the dual space \widehat{G}_6 of G_6 , i.e. we determine the boundaries of each orbit and we compute the limit sets of properly converging sequences of coadjoint orbits. In Section 5, we discover the continuity and infinity conditions and in Section 6, the most intricate one, we introduce the 6 different regions Γ_i of the dual space of G_6 according to the dimensions of the coadjoint orbits and obtain the boundary conditions for each of these regions. In the last section (Section 7), we describe the actual C^* -algebra of G_6 as an algebra of operator fields and we see that this C^* -algebra has the structure of an almost $C_0(\mathcal{K})$ - C^* -algebra.

2 A special class of C*-algebras

Definition 2.1. Let Γ be a topological Hausdorff space, let \mathcal{H} be a Hilbert space and denote by \mathcal{K} or $\mathcal{K}(\mathcal{H})$ the algebra of compact operators on \mathcal{H} . Let $C_0(\Gamma, \mathcal{K})$ be the space of all continuous mappings $\varphi : \Gamma \rightarrow \mathcal{K}(\mathcal{H})$ vanishing at infinity. Equipped with the norm

$$\|\varphi\|_\infty := \sup_{\gamma \in \Gamma} \|\varphi(\gamma)\|_{\text{op}} \quad \text{for } \varphi \in C_0(\Gamma, \mathcal{K}),$$

the space $C_0(\Gamma, \mathcal{K})$ is a C*-algebra, whose spectrum is homeomorphic to the topological space Γ .

Let now A be a separable C*-algebra and \widehat{A} be the unitary dual of A .

Definition 2.2. We suppose that there exists a finite increasing family $S_0 \subset S_1 \subset \dots \subset S_d = \widehat{A}$ of closed subsets of the spectrum \widehat{A} of A such that for $i = 1, \dots, d$, the subsets $\Gamma_0 = S_0$ and $\Gamma_i := S_i \setminus S_{i-1}$ are Hausdorff in their relative topologies. Furthermore we assume that for every $i \in \{0, \dots, d\}$ there exists a Hilbert space \mathcal{H}_i and a concrete realization $(\pi_\gamma, \mathcal{H}_i)$ of γ on the Hilbert space \mathcal{H}_i for every $\gamma \in \Gamma_i$. We also assume that the set S_0 is the collection \mathfrak{X} of all characters of A .

Definition 2.3. We define the Fourier transform $\mathcal{F} : A \rightarrow l^\infty(\widehat{A})$ as to be the mapping:

$$\mathcal{F}(a)(\gamma) = \widehat{a}(\gamma) := \pi_\gamma(a) \quad \text{for } \gamma \in \widehat{A}, a \in A.$$

For a subset $S \subset \widehat{A}$, denote by $CB(S)$ the *-algebra of all uniformly bounded operator fields $(\psi(\gamma) \in \mathcal{B}(\mathcal{H}_i))_{\gamma \in S \cap \Gamma_i, i=1, \dots, d}$, which are operator norm continuous on the subsets $\Gamma_i \cap S$ for every $i \in \{1, \dots, d\}$ for which $\Gamma_i \cap S \neq \emptyset$. We provide the algebra $CB(S)$ with the infinity-norm:

$$\|\psi\|_S := \sup_{\gamma \in S} \|\psi(\gamma)\|_{\text{op}}.$$

Definition 2.4. We say that a C*-algebra A is “almost $C_0(\mathcal{K})$ ” if for every $a \in A$:

1. The mappings $\gamma \mapsto \mathcal{F}(a)(\gamma)$ are norm-continuous on the different sets Γ_i . (We remark that for every closed subset S of \widehat{A} and every $a \in A$, the restriction $\mathcal{F}(a)|_S$ of the operator field $\mathcal{F}(a)$ to S is then contained in $CB(S)$).
2. For any $i = 1, \dots, d$, we have a sequence $(\sigma_{i,k} : CB(S_{i-1}) \rightarrow CB(S_i))_k$ of linear mappings which are uniformly bounded in k such that

$$\lim_{k \rightarrow \infty} \text{dis} \left((\sigma_{i,k}(\mathcal{F}(a)|_{S_{i-1}}) - \mathcal{F}(a)|_{\Gamma_i}), C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i)) \right) = 0$$

and such that

$$\lim_{k \rightarrow \infty} \text{dis} \left((\sigma_{i,k}(\mathcal{F}(a)|_{S_{i-1}})^* - \mathcal{F}(a^*)|_{\Gamma_i}), C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i)) \right) = 0.$$

(This condition justifies the name of “almost $C_0(\mathcal{K})$ ”).

Definition 2.5. Let $D^*(A)$ be the set of all operator fields φ defined over \widehat{A} such that

1. The field φ is uniformly bounded, i.e. $\|\varphi\| := \sup_{\gamma \in \widehat{A}} \|\varphi(\gamma)\|_{\text{op}} < \infty$.
2. $\varphi|_{\Gamma_i} \in CB(\Gamma_i)$ for every $i = 0, 1, \dots, d$.
3. For any sequence $(\gamma_k)_{k \in \mathbb{N}}$ going to infinity in \widehat{A} , we have $\lim_{k \rightarrow \infty} \|\varphi(\gamma_k)\|_{\text{op}} = 0$.
4. We have

$$\lim_{k \rightarrow \infty} \text{dis} \left((\sigma_{i,k}(\varphi|_{S_{i-1}}) - \varphi|_{\Gamma_i}), C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i)) \right) = 0$$

and

$$\lim_{k \rightarrow \infty} \text{dis} \left((\sigma_{i,k}(\varphi|_{S_{i-1}})^* - (\varphi|_{\Gamma_i})^*), C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i)) \right) = 0.$$

We see immediately that for every $a \in A$, the operator field $\mathcal{F}(a)$ is contained in the set $D^*(A)$. In fact it turns out that $D^*(A)$ is a C^* -subalgebra of $l^\infty(\widehat{A})$ and that A is isomorphic to $D^*(A)$.

Theorem 2.6. *Let A be a separable C^* -algebra which is almost $C_0(\mathcal{K})$. Then the subset $D^*(A)$ of the C^* -algebra $l^\infty(\widehat{A})$ is a C^* -subalgebra which is isomorphic to A under the Fourier transform.*

Proof. We remark that the conditions (1) to (4) imply that $D^*(A)$ is a closed involution-invariant subspace of $l^\infty(\widehat{A})$.

For $i = 0, \dots, d$, let D_i^* be the set of all operator fields defined over S_i satisfying conditions (1) to (4) on the sets S_j for $j = 1, \dots, i$. Then the sets D_i^* are closed subspaces of the C^* -algebra $l^\infty(S_i)$.

Let I_C be the closed two-sided ideal in A generated by the elements of the form $ab - ba, a, b \in A$. Then the space of characters $S_0 = \mathfrak{X}$ of A is the spectrum of A/I_C and D_0^* equals the algebra $C_0(S_0)$ of continuous functions on S_0 vanishing at infinity by the conditions (1) and (2). Since $\mathcal{F}(A)|_{S_0} = C_0(S_0)$ it follows that $D_0^* = \mathcal{F}(A)|_{S_0}$.

Let us assume now that for some $1 \leq i < d$, we have that $D_j^* = \mathcal{F}(A)|_{S_j}$ for $j = 0, \dots, i-1$. We shall prove then that $D_i^* = \mathcal{F}(A)|_{S_i}$. We know already that $\mathcal{F}(A)|_{S_i}$ is a subalgebra of the closed subspace $D_i^* \subset CB(S_i)$ and it follows from its definition that the restriction of D_i^* to S_{i-1} is contained in D_{i-1}^* . Hence

$$\mathcal{F}(A)|_{S_{i-1}} \subset D^*(A)|_{S_{i-1}} \subset D_{i-1}^* = \mathcal{F}(A)|_{S_{i-1}}.$$

Therefore $D^*(A)|_{S_{i-1}} = \mathcal{F}(A)|_{S_{i-1}}$. Let $\phi_1, \phi_2 \in D_i^*$. By our assumption, there exists $a_1, a_2 \in A$ such that $\phi_j|_{S_{i-1}} = \hat{a}_j|_{S_{i-1}}, j = 1, 2$. The product $\phi_1 \circ \phi_2$ satisfies also the conditions (1), (2) and (3). We shall show that it also satisfies the condition (4). We now have that

$$\begin{aligned} & \phi_1 \circ \phi_2|_{\Gamma_i} - \sigma_{i,k}(\phi_1 \circ \phi_2|_{S_{i-1}}) \\ &= \phi_1|_{\Gamma_i} \circ \phi_2|_{\Gamma_i} - \sigma_{i,k}(\hat{a}_1|_{S_{i-1}} \circ \hat{a}_2|_{S_{i-1}}) \\ &= \phi_1|_{\Gamma_i} \circ \phi_2|_{\Gamma_i} - \hat{a}_1|_{\Gamma_i} \circ \hat{a}_2|_{\Gamma_i} \\ & \quad + (\hat{a}_1|_{\Gamma_i} \circ \hat{a}_2|_{\Gamma_i}) - \sigma_{i,k}(\hat{a}_1|_{S_{i-1}} \circ \hat{a}_2|_{S_{i-1}}). \end{aligned}$$

Now

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \text{dis}((\hat{a}_1|_{\Gamma_i} \circ \hat{a}_2|_{\Gamma_i} - \sigma_{i,k}(\hat{a}_1|_{S_{i-1}} \circ \hat{a}_2|_{S_{i-1}})), C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i))) \\ &= \lim_{\delta \rightarrow 0} \text{dis}((\widehat{a_1 a_2}|_{\Gamma_i} - \sigma_{i,k}(\widehat{a_1 a_2}|_{S_{i-1}})), C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i))) \\ &= 0 \end{aligned}$$

by condition (2) in Definition 2.4. Furthermore,

$$\begin{aligned} & \phi_1|_{\Gamma_i} \circ \phi_2|_{\Gamma_i} - \hat{a}_1|_{\Gamma_i} \circ \hat{a}_2|_{\Gamma_i} \\ &= \phi_1|_{\Gamma_i} \circ (\phi_2|_{\Gamma_i} - \hat{a}_2|_{\Gamma_i}) + (\phi_1|_{\Gamma_i} - \hat{a}_1|_{\Gamma_i}) \circ \hat{a}_2|_{\Gamma_i} \\ &= \phi_1|_{\Gamma_i} \circ (\phi_2|_{\Gamma_i} - \sigma_{i,k}(\phi_2|_{S_{i-1}})) + \phi_1|_{\Gamma_i} \circ (\sigma_{i,k}(\hat{a}_2|_{S_{i-1}}) - \hat{a}_2|_{\Gamma_i}) \\ & \quad + (\phi_1|_{\Gamma_i} - \sigma_{i,k}(\phi_1|_{S_{i-1}})) \circ \hat{a}_2|_{\Gamma_i} + (\sigma_{i,k}(\hat{a}_1|_{S_{i-1}}) - \hat{a}_1|_{\Gamma_i}) \circ \hat{a}_2|_{\Gamma_i}. \end{aligned}$$

Hence,

$$\lim_{\delta \rightarrow 0} \text{dis}(\phi_1|_{\Gamma_i} \circ \phi_2|_{\Gamma_i} - \sigma_{i,k}(\phi_1 \circ \phi_2|_{S_{i-1}}), C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i))) = 0.$$

Therefore D_i^* is an algebra, i.e. a C*-subalgebra of $CB(S_i)$. The conditions (1) and (2) for the C*-algebra A tell us that the ideal $\ker(S_{i-1}) = \{a \in A : \hat{a}|_{S_{i-1}} = 0\}$ of A is isomorphic with the algebra $C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i)) \subset \mathcal{F}(A)$ of all continuous mappings defined on Γ_i with values in $\mathcal{K}(\mathcal{H}_i)$ and vanishing at infinity. Indeed, for any $\gamma \neq \gamma' \in S_i$, there exists an $a \in A$ such that $\gamma(a) \neq 0$, but $\gamma'(a) = 0$ and also $\hat{a}|_{S_{i-1}} = 0$. Then $\hat{a}|_{\Gamma_i}$ is in $C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i))$ by the condition (1),(2) and (3) and in this way we see that $\ker(S_{i-1})$ separates the points in Γ_i and that $\widehat{\ker(S_{i-1})}|_{\Gamma_i} \subset C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i))$. The theorem of Stone-Weierstrass says now that $\widehat{\ker(S_{i-1})}|_{\Gamma_i} = C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i))$.

Let now π be an element of the spectrum of D_i^* . Let $R_{i-1} : D_i^* \rightarrow D_{i-1}^*$ be the restriction map and let $I_{i-1} := \ker(R_{i-1}) \subset D_i^*$. Suppose that $\pi(I_{i-1}) = \{0\}$. We can then consider π as a representation of the algebra D_i^*/I_{i-1} . This algebra is isomorphic with the image of R_{i-1} which is itself isomorphic with $\mathcal{F}(A)|_{S_{i-1}}$ and therefore π is an element of S_{i-1} . Suppose that π is not trivial on I_{i-1} . By the condition (4) this kernel I_{i-1} is contained in $C_0(\Gamma_i, \mathcal{K})$ and since it contains $\ker(R_{i-1}) \cap \mathcal{F}(A)$, it is itself isomorphic to $C_0(\Gamma_i)$. Therefore π is an evaluation at some point in Γ_i .

We have seen that the spectrum of D_i^* coincides with the spectrum of the subalgebra $\mathcal{F}(A)_{|S_i}$. Hence by the theorem of Stone-Weierstrass (see [Di]), the C^* -algebra $\mathcal{F}(A)_{|S_i}$ and D_i^* coincide. \square

3 The group G_6

This paper aims to show that the C^* -algebra of the exponential Lie group $G_6 = \exp(\mathfrak{g}_6)$ is in the class of almost $C_0(\mathcal{K})$ - C^* -algebras.

Let $\mathfrak{g} = \mathfrak{g}_6$ be the Lie algebra spanned by the vectors A, B, P, Q, R, S and equipped with the Lie brackets:

$$\begin{aligned} [P, Q] &= R, & [P, R] &= S, & [A, P] &= \frac{1}{2}P, & [B, P] &= -\frac{1}{2}P, \\ [B, Q] &= Q, & [A, R] &= \frac{1}{2}R, & [B, R] &= \frac{1}{2}R, & [A, S] &= S, \\ [A, Q] &= 0, & [B, S] &= 0. \end{aligned}$$

We introduce the following closed subgroups and coordinate functions on G , write

$$\begin{aligned} \mathfrak{g}_0 &:= \text{span}\{A, B, P\}, & G_0 &:= \exp(\mathfrak{g}_0), \\ \mathfrak{h} &:= \text{span}\{Q, R, S\}, & H &:= \exp(\mathfrak{h}). \end{aligned} \tag{3.0.1}$$

Then the subset G_0 is a closed subgroup of G , and the subset H is a closed normal subgroup. We have

$$G = G_0 \cdot H$$

as a topological product. For $(a, b, p, q, r, s) \in \mathbb{R}^6$, let

$$g = E(a, b, p, q, r, s) := \exp(aA) \exp(bB) \exp(pP) \exp(qQ) \exp(rR) \exp(sS).$$

3.1 A parametrization of the set of coadjoint orbits

We give in this section a system of representatives of the set of coadjoint orbits $\mathfrak{g}^*/\text{Ad}^*G$.

Let $\{A^*, B^*, P^*, Q^*, R^*, S^*\}$ be the dual basis of $\{A, B, P, Q, R, S\}$, and $f = a^*A^* + b^*B^* + p^*P^* + q^*Q^* + r^*R^* + s^*S^* \in \mathfrak{g}^*$. Then

$$\begin{aligned} \text{Ad}^*(g)(f) &= \left(a^* + \frac{p}{2}p^* + \frac{1}{2}(pq + r)r^* + \left(s + \frac{pr}{2} \right) s^* \right) A^* \\ &\quad + \left(b^* - \frac{p}{2}p^* + qq^* + \frac{1}{2}(r - pq)r^* - \frac{pr}{2}s^* \right) B^* \\ &\quad + e^{\frac{-a+b}{2}}(p^* + qr^* + rs^*)P^* \\ &\quad + e^{-b} \left(q^* - pr^* + \frac{1}{2}p^2s^* \right) Q^* + e^{\frac{-a-b}{2}}(r^* - ps^*)R^* + e^{-a}s^*S^*. \end{aligned} \tag{3.1.1}$$

In the sequel, we write $\mathbb{R}^+ := \{x \in \mathbb{R}; x > 0\}$ and $\mathbb{R}^{+,0} := \mathbb{R}^+ \cup \{0\}$. We note that the determinant of the matrix $(f([V, W]))_{V, W \in \{A, B, P, Q, R, S\}}$ is $\frac{1}{4}s^{*2}(2q^*s^* - r^{*2})^2$, and there are open orbits. The following list gives a parametrization of the coadjoint orbits $\mathfrak{g}^*/\text{Ad}^*G$.

0-dimensional orbits $a^*A^* + b^*B^*$, where $a^*, b^* \in \mathbb{R}$.

2-dimensional orbits

(2d-1) $\alpha^*(\frac{A^*+B^*}{2}) + \varepsilon P^*$, where $\alpha^* \in \mathbb{R}$, $\varepsilon = \pm 1$,

$$\Omega_{\alpha^*(\frac{A^*+B^*}{2})+\varepsilon P^*} = \{(\frac{\alpha^*}{2} + \frac{p}{2}\varepsilon)A^* + (\frac{\alpha^*}{2} - \frac{p}{2}\varepsilon)B^* + e^{-\alpha}\varepsilon P^*; \alpha, p \in \mathbb{R}\}.$$

[see (3.1.11), (3.1.12) below]

(2d-2) $a^*A^* + \nu Q^*$, where $a^* \in \mathbb{R}$, $\nu = \pm 1$,

$$\Omega_{a^*A^*+\nu Q^*} = \{a^*A^* + q\nu B^* + e^{-b}\nu Q^*; b, q \in \mathbb{R}\}.$$

[see (3.1.10)]

4-dimensional orbits

(4d-1) $\varepsilon P^* + \nu Q^*$, where $\varepsilon = \pm 1$, $\nu = \pm 1$,

$$\Omega_{\varepsilon P^*+\nu Q^*} = \{pA^* + qB^* + e^{\frac{-a+b}{2}}(\varepsilon)P^* + e^{-b}\nu Q^*; p, q, a, b \in \mathbb{R}\}.$$

[see (3.1.9)]

(4d-2) $b^*B^* + \varepsilon R^*$, where $b^* \in \mathbb{R}$, $\varepsilon = \pm 1$,

$$\Omega_{b^*B^*+\varepsilon R^*} = \{rA^* + (b^* + r - pq\varepsilon)B^* + qP^* + (-pe^{-a})Q^* + e^{-a}\varepsilon R^*; a, p, q, r \in \mathbb{R}\}.$$

[see (3.1.7), (3.1.8)]

(4d-3) $b^*B^* + \varepsilon S^*$, where $b^* \in \mathbb{R}$, $\varepsilon = \pm 1$,

$$\Omega_{b^*B^*+\varepsilon S^*} = \{sA^* + (b^* - \frac{pr}{2}\varepsilon)B^* + e^{\frac{-a}{2}}(r\varepsilon)P^* + (\frac{1}{2}p^2\varepsilon)Q^* + e^{\frac{-a}{2}}(-p\varepsilon)R^* + e^{-a}\varepsilon S^*; \\ s, p, r, a \in \mathbb{R}\}.$$

[see (3.1.4), (3.1.5)]

6-dimensional orbits (6d) $\nu Q^* + \varepsilon S^*$, where $\nu = \pm 1$, $\varepsilon = \pm 1$,

$$\Omega_{\nu Q^*+\varepsilon S^*} = \{sA^* + rB^* + qP^* + e^{-b}(\nu + \frac{p^2\varepsilon}{2})Q^* + (-e^{-\frac{a+b}{2}}p\varepsilon)R^* + (e^{-a}\varepsilon)S^*; \\ s, r, q, a, b, p \in \mathbb{R}\}.$$

[see (3.1.2), (3.1.3)]

Proof and description of each orbit. Let Ω be a coadjoint orbit and $f \in \Omega$.

Case I) Suppose $f(S) \neq 0$. Then by (3.1.1), there exists $g = E(a, 0, p, 0, r, s)$ such that $g \cdot f = b^*B^* + q^*Q^* + \varepsilon S^*$, where $\varepsilon = \pm 1$ and $b^*, q^* \in \mathbb{R}$. Thus let $f = b^*B^* + q^*Q^* + \varepsilon S^*$.

I-1) If $q^* \neq 0$, then there exists $g = E(0, b, 0, q, 0, 0)$ such that $\text{Ad}^*g(f) = (b^* + qq^*)B^* + e^{-b}q^*Q^* + \varepsilon S^* = \nu Q^* + \varepsilon S^*$, where $\nu = \pm 1$.

$$\begin{aligned} f : &= \nu Q^* + \varepsilon S^*, \\ \text{Ad}^*E(a, b, p, q, r, s)(f) &= \bar{a}A^* + \bar{b}B^* + \bar{p}P^* + \bar{q}Q^* + \bar{r}R^* + \bar{s}S^*, \end{aligned} \quad (3.1.2)$$

where

$$\begin{aligned} \bar{a} &= \left(s + \frac{1}{2}pr\right)\varepsilon, \quad \bar{b} = \left(q\nu - \frac{1}{2}pr\varepsilon\right), \quad \bar{p} = e^{-\frac{a+b}{2}}r\varepsilon, \\ \bar{q} &= e^{-b}\left(\nu + \frac{1}{2}p^2\varepsilon\right), \quad \bar{r} = -e^{-\frac{a-b}{2}}p\varepsilon, \quad \bar{s} = e^{-a}\varepsilon. \end{aligned}$$

Let $\Phi(l) := 2l(Q)l(S) - l(R)^2$ for $l \in \mathfrak{g}^*$. Then $\Phi(\bar{f}) = 2\bar{q}\bar{s} - \bar{r}^2 = 2e^{-a-b}\nu\varepsilon \neq 0$, and we have

$$\Omega_f = \{l; \Phi(l) \neq 0, l(S) \neq 0\}. \quad (3.1.3)$$

I-2) If $q^* = 0$, then $f = b^*B^* + \varepsilon S^*$ and $\mathfrak{g}(f) = \mathbb{R}\text{-span}\{B, Q\}$.

$$\text{Ad}^*E(a, 0, p, 0, r, s)(f) = \bar{a}A^* + \bar{b}B^* + \bar{p}P^* + \bar{q}Q^* + \bar{r}R^* + \bar{s}S^*, \quad (3.1.4)$$

where

$$\begin{aligned} \bar{a} &= \left(s + \frac{pr}{2}\right)\varepsilon, \quad \bar{b} = \left(b^* - \frac{pr}{2}\varepsilon\right), \quad \bar{p} = re^{-\frac{a}{2}}\varepsilon, \\ \bar{q} &= \frac{1}{2}p^2\varepsilon, \quad \bar{r} = -pe^{-\frac{a}{2}}\varepsilon, \quad \bar{s} = e^{-a}\varepsilon. \end{aligned}$$

Let Φ, Φ_2 be functions on \mathfrak{g}^* defined by

$$\Phi(l) = 2l(Q)l(S) - l(R)^2, \quad \Phi_2(l) = l(R)l(P) - 2(l(B) - b^*)l(S).$$

Then we have

$$\Omega_f = \Phi^{-1}(0) \cap \Phi_2^{-1}(0) \cap (\mathbb{R}A^* + \mathbb{R}B^* + \mathbb{R}P^* + (\mathbb{R}^{+,0})\varepsilon Q^* + \mathbb{R}R^* + \mathbb{R}^+\varepsilon S^*). \quad (3.1.5)$$

In fact, let $\bar{f} := (\bar{a}, \bar{b}, \bar{p}, \bar{q}, \bar{r}, \bar{s})$ be an element of the right hand side. We take

$$a := -\log(\varepsilon\bar{s}), \quad p := -\bar{r}e^{\frac{a}{2}}\varepsilon, \quad r := \bar{p}e^{-\frac{a}{2}}\varepsilon, \quad s := \bar{a} - \frac{pr}{2}.$$

Then we have $\text{Ad}^*E(a, 0, p, 0, r, s)f = \bar{f}$, this is, $\bar{f} \in \Omega_f$. The reverse inclusion is easily verified.

Case II) Suppose $f(S) = 0$. Then $\text{Ad}^*G(f)(S) = \{0\}$. Then (3.1.1) is reduced to

$$\begin{aligned} \text{Ad}^*g(f) &= \left(a^* + \frac{p}{2}p^* + \frac{1}{2}(pq + r)r^*\right)A^* \\ &\quad + \left(b^* - \frac{p}{2}p^* + qq^* + \frac{1}{2}(r - pq)r^*\right)B^* \\ &\quad + e^{\frac{-a+b}{2}}(p^* + qr^*)P^* + e^{-b}(q^* - pr^*)Q^* + e^{\frac{-a-b}{2}}r^*R^*. \end{aligned} \quad (3.1.6)$$

II-1) Suppose $f(R) = r^* \neq 0$. Then by (3.1.6), there exists $g = E(\alpha, \alpha, p, q, r, 0)$ such that $\text{Ad}^*g(f) = b^*B^* + \varepsilon R^*$, where $\varepsilon = \pm 1$, $b^* \in \mathbb{R}$. Let $f = b^*B^* + \varepsilon R^*$. Then $\mathfrak{g}(f) = \mathbb{R}\text{-span}\{S, A - B\}$. Let $X = A + B$, $Y = A - B$, $X^* = \frac{1}{2}(A^* + B^*)$, $Y^* = \frac{1}{2}(A^* - B^*)$. Then $\{X^*, Y^*, P^*, Q^*, R^*, S^*\}$ is the dual basis of $\{X, Y, P, Q, R, S\}$.

$$\begin{aligned} f &:= b^*B^* + \varepsilon R^*, \\ \text{Ad}^*E(\alpha, \alpha, p, q, r, 0)(f) &= \frac{1}{2}(pq + r)\varepsilon A^* + \left(b^* + \frac{1}{2}(r - pq)\varepsilon\right)B^* \\ &\quad + q\varepsilon P^* - e^{-\alpha}p\varepsilon Q^* + e^{-\alpha}\varepsilon R^* \\ &= (b^* + r\varepsilon)X^* + (-b^* + pq\varepsilon)Y^* + q\varepsilon P^* - e^{-\alpha}p\varepsilon Q^* + e^{-\alpha}\varepsilon R^*, \end{aligned} \quad (3.1.7)$$

$$\Omega_f = \{\bar{x}X^* + \bar{y}Y^* + \bar{p}P^* + \bar{q}Q^* + \bar{r}R^*; \bar{r} \in \varepsilon\mathbb{R}^+, \bar{y} = -b^* - (\bar{p}\bar{q}/\bar{r})\} \quad (3.1.8)$$

II-2) Suppose $f(R) = 0$. Then $G \cdot f(R) = \{0\}$, and we have

$$\text{Ad}^*g(f) = \left(a^* + \frac{p}{2}p^*\right)A^* + \left(b^* - \frac{p}{2}p^* + qq^*\right)B^* + e^{\frac{-a+b}{2}}p^*P^* + e^{-b}q^*Q^*.$$

II-2-i) Suppose $q^* \neq 0$ and $p^* \neq 0$. Then by (3.1.9), there exists $g = E(a, b, p, q, 0, 0)$ such that $\text{Ad}^*g(f) = \varepsilon P^* + \nu Q^*$, where $\varepsilon = \pm 1, \nu = \pm 1$. We have $\mathfrak{g}(\varepsilon P^* + \nu Q^*) = \mathbb{R}\text{-span}\{S, R\}$, and

$$\begin{aligned} f &:= \varepsilon P^* + \nu Q^*, \\ \text{Ad}^*E(a, b, p, q, 0, 0)(f) &= \frac{p}{2}\varepsilon A^* + \left(-\frac{p}{2}\varepsilon + q\nu\right)B^* + e^{\frac{-a+b}{2}}\varepsilon P^* + e^{-b}\nu Q^*, \\ \Omega_f &= \mathbb{R}A^* + \mathbb{R}B^* + \mathbb{R}^+\varepsilon P^* + \mathbb{R}^+\nu Q^*. \end{aligned} \quad (3.1.9)$$

II-2-ii) Suppose $q^* \neq 0$ and $p^* = 0$. Then there exists $g = E(0, b, 0, q, 0, 0)$ such that $g \cdot f = a^*A^* + \nu Q^*$. We have $\mathfrak{g}(a^*A^* + \nu Q^*) = \mathbb{R}\text{-span}\{A, P, R, S\}$,

$$\begin{aligned} f &:= a^*A^* + \nu Q^*, \\ \text{Ad}^*E(0, b, 0, q, 0, 0)(f) &= a^*A^* + q\nu B^* + e^{-b}\nu Q^*, \\ \Omega_f &= a^*A^* + \mathbb{R}B^* + \mathbb{R}^+\nu Q^*. \end{aligned} \quad (3.1.10)$$

II-2-iii) Suppose $q^* = 0$ and $p^* \neq 0$. Then there exists $g = \exp(\alpha(A - B)) \exp(pP)$ such that $g \cdot f = \alpha^*(\frac{A^*+B^*}{2}) + \varepsilon P^*$, where $\alpha^* \in \mathbb{R}$, $\varepsilon = \pm 1$. Let $f = \alpha^*(\frac{A^*+B^*}{2}) + \varepsilon P^*$. Then we have $\mathfrak{g}(f) = \mathbb{R}\text{-span}\{A + B, R, S, Q\}$. Letting $X = A + B$, $Y = A - B$, $X^* = \frac{A^*+B^*}{2}$, $Y^* = \frac{A^*-B^*}{2}$, we have

$$f := \alpha^*(\frac{A^*+B^*}{2}) + \varepsilon P^* = \alpha^* X^* + \varepsilon P^*, \quad (3.1.11)$$

$$\begin{aligned} \text{Ad}^* E(\alpha, -\alpha, p, 0, 0, 0)(f) &= \left(\frac{\alpha^*}{2} + \frac{p}{2}\varepsilon\right) A^* + \left(\frac{\alpha^*}{2} - \frac{p}{2}\varepsilon\right) B^* + e^{-\alpha}\varepsilon P^* \\ &= \alpha^* X^* + p\varepsilon Y^* + e^{-\alpha}\varepsilon P^*, \end{aligned}$$

$$\Omega_f = \alpha^* X^* + \mathbb{R}Y^* + \mathbb{R}^+\varepsilon P^*. \quad (3.1.12)$$

II-2-iv) Suppose $q^* = p^* = 0$. Then $G \cdot f = \{a^* A^* + b^* B^*\}$ which is a point. \square

4 The topology of \widehat{G}

4.1 The closure of the different orbits and their corresponding irreducible representations

We shall need in our description of $C^*(G)$ the following criterion for the compactness of an operator $\pi(F)$, $F \in A$, where π is an irreducible representation of a type I C^* -algebra A .

Theorem 4.1. *The operator $\pi(F)$, $F \in A$, is compact if and only if $\pi'(F) = 0$ for every π' in the boundary of π .*

Since the topology of the spectrum $\widehat{C^*(G)}$ of an exponential Lie group $G = \exp(\mathfrak{g})$ is that of the topology of the space of coadjoint orbits \mathfrak{g}^*/G according to [Lep-Lud], we have the following.

Theorem 4.2. *If $G = \exp(\mathfrak{g})$ is an exponential Lie group, ℓ is an element of \mathfrak{g}^* and π_ℓ is the corresponding irreducible unitary representation, then for $F \in C^*(G)$ the operator $\pi_\ell(F)$ is compact if and only if $\pi_q(F) = 0$ for any q in the boundary of the orbit Ω_ℓ of ℓ .*

We are therefore forced to determine the boundary of every coadjoint orbit listed in Subsection 3.1 of our group G .

4.1.1 The open orbits of $\Omega_{\varepsilon S^* \pm \varepsilon Q^*}$, $\varepsilon = \pm 1$

Let $f = \varepsilon S^* + \nu Q^*$, where $\nu = \pm \varepsilon$. Then $\text{Ad}^* G(f)$ is an open orbit, and we have

$$\Omega_f = \{l; \Phi(l) \neq 0, l(S) \neq 0\},$$

where $\Phi(l) := 2l(Q)l(S) - l(R)^2$ for $l \in \mathfrak{g}^*$. Noting that $\Phi(l) \leq 0$ if $l(S) = 0$, we have

$$\overline{\Omega_f} \setminus \Omega_f = \begin{cases} \{l; \Phi(l) = 0, l(S) \geq 0\} & \varepsilon = 1, \nu = 1, \\ \{l; \Phi(l) \leq 0, l(S) = 0\} \cup \{l; \Phi(l) = 0, l(S) \geq 0\} & \varepsilon = 1, \nu = -1, \\ \{l; \Phi(l) \leq 0, l(S) = 0\} \cup \{l; \Phi(l) = 0, l(S) \leq 0\} & \varepsilon = -1, \nu = 1, \\ \{l; \Phi(l) = 0, l(S) \leq 0\} & \varepsilon = -1, \nu = -1. \end{cases}$$

Considering the natural projection $\mathfrak{g}^* \rightarrow \mathfrak{h}^*$ defined by restriction and denoting $G \cdot l := \text{Ad}^* G(l)|_{\mathfrak{h}}$ for $l \in \mathfrak{g}^*$, we have

$$(\overline{\Omega_f}|_{\mathfrak{h}}) \setminus (\Omega_f|_{\mathfrak{h}}) = \begin{cases} \{G \cdot S^*, G \cdot Q^*, 0\} & \varepsilon = 1, \nu = 1, \\ \{G \cdot S^*, \pm G \cdot Q^*, \pm G \cdot R^*, 0\} & \varepsilon = 1, \nu = -1, \\ \{-G \cdot S^*, \pm G \cdot Q^*, \pm G \cdot R^*, 0\} & \varepsilon = -1, \nu = 1, \\ \{-G \cdot S^*, -G \cdot Q^*, 0\} & \varepsilon = -1, \nu = -1. \end{cases}$$

Realization of $\pi_f \in \widehat{G}$: We realize π_f by taking the Pukanszky polarization $\mathfrak{h} = \mathbb{R}\text{-span}\{Q, R, S\}$ on $L^2(\exp(\mathbb{R}A + \mathbb{R}B)\exp(\mathbb{R}P))$. We note that $\mathfrak{g}_0 := \mathbb{R}\text{-span}\{A, B, P\}$ is a subalgebra, and $G = G_0H$, where $G_0 := \exp \mathfrak{g}_0$. Writing $g_0 = g_0(a, b, p) := \exp(aA + bB)\exp(pP)$, $g_1 = g_1(q, r, s) = \exp(qQ + rR + sS)$, $h = h(t, u, v) = \exp(tA + uB)\exp vP$, we have

$$\pi_f(g_0g_1)\xi(h) = \chi_f(h^{-1}g_0g_1g_0^{-1}h)\xi(g_0^{-1}h) = \chi_{\text{Ad}^*(g_0^{-1}h)(f)}(g_1)\xi(g_0^{-1}h).$$

We take the left Haar measure dg_0 , dg_1 , and dg on G_0 , H , and G , respectively, which are defined by transferring Lebesgue measures by $(a, b, p) \in \mathbb{R}^3 \mapsto g_0(a, b, p) \in G_0$, $(q, r, s) \in \mathbb{R}^3 \mapsto g_1(q, r, s) \in H$, and $dg = dg_0dg_1$. Let $F \in L^1(G)$. For $l \in \mathfrak{h}^*$ we denote

$$\widehat{F}^{\mathfrak{h}}(a, b, p, l) := \widehat{F}^{\mathfrak{h}}(g_0(a, b, p))(l) := \int_{\mathfrak{h}} F(g_0g_1)\chi_l(g_1)dg_1, \quad a, b, p \in \mathbb{R}.$$

Then we have

$$\begin{aligned} \pi_f(F)\xi(h) &= \int_{G_0H} \pi_f(g_0g_1)\xi(h)F(g_0g_1)dg_0dg_1 \\ &:= \int_{\mathbb{R}^6} F(g_0(a, b, p)g_1(q, r, s))\pi_f(g_0(a, b, p)g_1(q, r, s))\xi(h)dadbpdqdrds \\ &= \int_{G_0H} F(g_0g_1)\chi_{\text{Ad}^*(g_0^{-1}h)(f)}(g_1)\xi(g_0^{-1}h)dg_0dg_1 \\ &= \int_{G_0H} F(hg_0g_1)\chi_{\text{Ad}^*(g_0^{-1})(f)}(g_1)\xi(g_0^{-1})dg_0dg_1 \\ &= \int_{G_0H} F(hg_0^{-1}g_1)\chi_{\text{Ad}^*g_0(f)}(g_1)\xi(g_0)\Delta_{G_0}^{-1}(g_0)dg_0dg_1 \\ &= \int_{G_0} \widehat{F}^{\mathfrak{h}}(hg_0^{-1})(\text{Ad}^*g_0(f)|_{\mathfrak{h}})\Delta_{G_0}^{-1}(g_0)\xi(g_0)dg_0 \\ &=: \int_{G_0} \mathcal{K}_F(h, g_0)\xi(g_0)dg_0, \end{aligned}$$

where Δ_{G_0} is the modular function of G_0 and

$$\begin{aligned}
\mathcal{K}_F(h, g_0) &= \mathcal{K}_F(h(t, u, v), g_0(a, b, p)) \\
&= \widehat{F}^{\mathfrak{h}}(hg_0^{-1})(\text{Ad}^*g_0(f|_{\mathfrak{h}}))\Delta_{G_0}^{-1}(g_0) \\
&= \widehat{F}^{\mathfrak{h}}(g_0(t-a, u-b, e^{\frac{a-b}{2}}(v-p))(\text{Ad}^*g_0(a, b, p)(f|_{\mathfrak{h}}))e^{\frac{a-b}{2}} \\
&= \widehat{F}^{\mathfrak{h}}(t-a, u-b, e^{\frac{a-b}{2}}(v-p), \varepsilon((e^{-b}(\pm 1 + \frac{p^2}{2})Q^* + (-e^{-\frac{a+b}{2}}p)R^* + e^{-a}S^*))e^{\frac{1}{2}(a-b)}.
\end{aligned} \tag{4.1.1}$$

4.1.2 The orbits of $\Omega_{b^*B^*+\varepsilon S^*}$, $\varepsilon = \pm 1, b^* \in \mathbb{R}$

Let $f = b^*B^* + \varepsilon S^*$. Then $\mathfrak{g}(f) = \mathbb{R}\text{-span}\{B, Q\}$. We have

$$\Omega_f = \Phi^{-1}(0) \cap \Phi_2^{-1}(0) \cap (\mathbb{R}A^* + \mathbb{R}B^* + \mathbb{R}P^* + (\varepsilon\mathbb{R}^{+,0})Q^* + \mathbb{R}R^* + \varepsilon\mathbb{R}^+S^*),$$

where

$$\Phi(l) = 2l(Q)l(S) - l(R)^2, \quad \Phi_2(l) = l(R)l(P) - 2(l(B) - b^*)l(S).$$

Then

$$\begin{aligned}
\overline{\Omega_f} \setminus \Omega_f &= (\mathbb{R}A^* + \mathbb{R}B^* + \varepsilon\mathbb{R}^+Q^*) \cup (\mathbb{R}A^* + \mathbb{R}B^* + \mathbb{R}P^*) \\
&= \bigcup_{a^* \in \mathbb{R}} \text{Ad}^*G(a^*A^* + \varepsilon Q^*) \cup \bigcup_{x^* \in \mathbb{R}, \nu = \pm 1} \text{Ad}^*G(x^*X^* + \nu P^*) \\
&\quad \bigcup_{a^*, \lambda \in \mathbb{R}} (a^*A^* + \lambda B^*).
\end{aligned} \tag{4.1.2}$$

Proof of (4.1.2): Since

$$\overline{\Omega_f} \subset \Phi^{-1}(0) \cap \Phi_2^{-1}(0) \cap (\mathbb{R}A^* + \mathbb{R}B^* + \mathbb{R}P^* + \mathbb{R}^{+,0}\varepsilon Q^* + \mathbb{R}R^* + \mathbb{R}^{+,0}\varepsilon S^*),$$

we first note that

$$\overline{\Omega_f} \setminus \Omega_f \subset \Phi^{-1}(0) \cap \Phi_2^{-1}(0) \cap (\mathbb{R}A^* + \mathbb{R}B^* + \mathbb{R}P^* + \mathbb{R}^{+,0}\varepsilon Q^* + \mathbb{R}R^*).$$

In fact, let $l \in \overline{\Omega_f}$, and suppose $l(S) \neq 0$. Then there exists $g \in G$ such that $\text{Ad}^*g(l) = b'B^* + q'Q^* + \varepsilon S^*$. Since $\overline{\Omega_f}$ is Ad^*G -invariant, we have $\Phi(\text{Ad}^*g(l)) = 2q'\varepsilon = 0$ and $\Phi_2(\text{Ad}^*g(l)) = -2(b' - b^*)\varepsilon = 0$, thus we have $\text{Ad}^*g(l) = f$, this is, $l \in \Omega_f$.

Let $f_\infty = \bar{a}A^* + \bar{b}B^* + \bar{p}P^* + \bar{q}Q^* + \bar{r}R^* \in \overline{\Omega_f} \setminus \Omega_f$. Then $\bar{r} = 0$, since $\Phi(f_\infty) = 0$. Suppose $\bar{f} = \lim_{k \rightarrow \infty} \text{Ad}^*E(a_k, 0, p_k, 0, r_k, s_k)(f)$. Since $\bar{f}(S) = \lim_k e^{-a_k}\varepsilon = 0$, we have $\lim_k a_k = \infty$.

Suppose $\bar{q} = \lim_k \frac{1}{2}p_k^2\varepsilon \neq 0$, since $\bar{b} = \lim_k (b^* - \frac{p_k r_k}{2}\varepsilon)$, we have that the sequence $\{r_k\}$ is bounded and $\bar{p} = \lim_k r_k e^{-\frac{a_k}{2}}\varepsilon = 0$. It follows that $\overline{\Omega_f} \setminus \Omega_f \subset (\mathbb{R}A^* + \mathbb{R}B^* + \varepsilon\mathbb{R}^+Q^*) \cup (\mathbb{R}A^* + \mathbb{R}B^* + \mathbb{R}P^*)$.

Conversely, let $\bar{f} = \bar{a}A^* + \bar{b}B^* + \bar{q}Q^*$ with $\bar{q} \in \varepsilon\mathbb{R}^+$. When $a \rightarrow \infty, p \rightarrow \sqrt{2\bar{q}\varepsilon}$, $r \rightarrow 2\varepsilon(b^* - \bar{b})\sqrt{2\bar{q}\varepsilon}^{-1}$ and $s \rightarrow \varepsilon(\bar{a} + \bar{b} - b^*)$, we have $\text{Ad}^*E(a, 0, p, 0, r, s)f \rightarrow \bar{f}$. Suppose $\bar{f} = \bar{a}A^* + \bar{b}B^* + \bar{p}P^*$ with $\bar{p} \neq 0$, and let $p(a) := 2(b^* - \bar{b})\bar{p}^{-1}e^{-\frac{a}{2}}$, $r(a) := \varepsilon\bar{p}e^{\frac{a}{2}}$, and $s(a) := \varepsilon\bar{a} - \frac{p(a)r(a)}{2}$. Then $\text{Ad}^*E(a, 0, p(a), 0, r(a), s(a))f \rightarrow$

\bar{f} as $a \rightarrow \infty$. Suppose $\bar{f} = \bar{a}A^* + \bar{b}B^*$. Let $p(a) := 2(b^* - \bar{b})\varepsilon e^{-\frac{a}{4}}$, $r(a) := e^{\frac{a}{4}}$, $s(a) := \varepsilon\bar{a} - \frac{p(a)r(a)}{2}$. Then $\text{Ad}^*E(a, 0, p(a), 0, r(a), s(a))f \rightarrow \bar{f}$ as $a \rightarrow \infty$. \square

Realization of $\pi_f \in \widehat{G}$: Taking the Pukanszky polarization $\mathfrak{b} = \mathbb{R}\text{-span}\{B, Q, R, S\}$ and $F \in L^1(G)$, we realize π_f and $\pi_f(F)$ on $L^2(\exp(\mathbb{R}X + \mathbb{R}P))$, where $X = A + B$, as follows: Let $E_0(x, p) := \exp(xX + pP)$,

$$h = h(b, q, r, s) := \exp(bB) \exp(qQ + rR + sS) \in \exp(\mathfrak{b}), \text{ and}$$

$$E_0(x, p)_b := \exp(bB)E_0(x, p) \exp(-bB) \in \exp(\mathbb{R}X + \mathbb{R}P).$$

For $\xi \in L^2(\exp(\mathbb{R}X + \mathbb{R}P))$, we have

$$\begin{aligned} & \pi_f(E_0(x, p)h(b, q, r, s))\xi(E_0(t, u)) \\ &= \chi(F)(E_0(t - x, u - p)_{(-b)}^{-1}h(0, q, r, s)E_0(t - x, u - p)_{(-b)})\chi_f(\exp(bB)) \\ & \quad \left(\frac{\Delta_{\exp(\mathfrak{b})}}{\Delta_G}(\exp(-bB)) \right)^{\frac{1}{2}} \xi(E_0(t - x, u - p)_{(-b)}) \\ &= \chi \text{Ad}^*E_0(t - x, u - p)_{(-b)}(f)(h(0, q, r, s))\chi_f(\exp(bB)) \left(\frac{\Delta_{\exp(\mathfrak{b})}}{\Delta_G}(\exp(-bB)) \right)^{\frac{1}{2}} \xi(E_0(t - x, u - p)_{(-b)}) \\ &= \exp \left(i\varepsilon e^{-t+x} \left(\frac{1}{2}q(u - p)^2 e^b - r(u - p)e^{\frac{b}{2}} + s \right) + ib^*b + \frac{b}{4} \right) \xi(t - x, e^{\frac{b}{2}}(u - p)). \end{aligned}$$

We denote $\widehat{F}^{\mathfrak{h}}(g)(l) = \int_{\mathfrak{h}} \chi_l(\exp(qQ + rR + sS))F(g \exp(qQ + rR + sS))dqdrds$ for $l \in \mathfrak{h}^*$ satisfying $l([\mathfrak{h}, \mathfrak{h}]) = \{0\}$, and for integrable functions F as before. We take the left Haar measure dg on G defined by

$$\int_G F(g)dg := \int_{\mathbb{R}^6} F(E_0(x, p)h(b, q, r, s))e^{\frac{b}{2}}dx dp db dq dr ds.$$

Then we have

$$\begin{aligned} & \pi_f(F)\xi(E_0(t, u)) = \int_{\mathbb{R}^6} F(E_0(x, p)h(b, q, r, s))\pi_f(E_0(x, p)h(b, q, r, s))\xi(E_0(t, u))e^{\frac{b}{2}}dx dp db dq dr ds \\ &= \int_{\mathbb{R}^6} F(E_0(x, p)h(b, q, r, s))\chi \text{Ad}^*E_0(t - x, u - p)_{(-b)}(f)(h(0, q, r, s))\chi_f(\exp(bB)) \\ & \quad \left(\frac{\Delta_{\exp(\mathfrak{b})}}{\Delta_G}(\exp(-bB)) \right)^{\frac{1}{2}} \xi(E_0(t - x, u - p)_{(-b)})e^{\frac{b}{2}}dx dp db dq dr ds \\ &= \int_{\mathbb{R}^3} \widehat{F}^{\mathfrak{h}}(E_0(t - x, u - p) \exp(bB)) (\text{Ad}^*E_0(x, p)_{(-b)}(f|_{\mathfrak{h}})) \chi_f(\exp(bB)) \\ & \quad \left(\frac{\Delta_{\exp(\mathfrak{b})}}{\Delta_G}(\exp(-bB)) \right)^{\frac{1}{2}} \xi(E_0(x, p)_{(-b)})e^{\frac{b}{2}}dx dp db \\ &= \int_{\mathbb{R}^3} \widehat{F}^{\mathfrak{h}}(E_0(t - x, u - e^{-\frac{b}{2}}p) \exp(bB)) (\text{Ad}^*E_0(x, p)(f|_{\mathfrak{h}})) \chi_f(\exp(bB))\xi(E_0(x, p))e^{\frac{b}{4}}dx dp db \\ &= \int_{\mathbb{R}^3} \widehat{F}^{\mathfrak{h}}(E_0(t - x, u - e^{-\frac{b}{2}}p) \exp(bB)) \left(\varepsilon e^{-x} \left(\frac{1}{2}p^2 Q^* - pR^* + S^* \right) \right) e^{ib^*b} e^{\frac{b}{4}} \xi(E_0(x, p))dx dp db \\ &=: \int_{\mathbb{R}^2} \mathcal{K}_F(t, u; x, p)\xi(E_0(x, p))dx dp, \end{aligned}$$

where

$$\begin{aligned} & \mathcal{K}_F(t, u; x, p) \\ &:= \int_{\mathbb{R}} \widehat{F^b}(E_0(t-x, u - e^{-\frac{b}{2}}p) \exp(bB)) \left(\varepsilon e^{-x} \left(\frac{1}{2} p^2 Q^* - pR^* + S^* \right) \right) e^{ib^*b} e^{\frac{b}{4}} db. \end{aligned} \quad (4.1.3)$$

4.1.3 The orbits of $b^*B^* + \varepsilon R^*, b^* \in \mathbb{R}, \varepsilon = \pm 1$

Let $f = b^*B^* + \varepsilon R^*$. Then $\mathfrak{g}(f) = \mathbb{R}\text{-span}\{S, A - B\}$. Writing $X = A + B$, $Y = A - B$, $X^* = \frac{1}{2}(A^* + B^*)$, $Y^* = \frac{1}{2}(A^* - B^*)$, we have

$$\Omega_f = \{\bar{x}X^* + \bar{y}Y^* + \bar{p}P^* + \bar{q}Q^* + \bar{r}R^*; \bar{r} \in \varepsilon\mathbb{R}^+, \bar{y} = -b^* - (\bar{p}\bar{q}/\bar{r})\},$$

$$\begin{aligned} \overline{\Omega_f} \setminus \Omega_f &= \{\bar{x}X^* + \bar{y}Y^* + \bar{p}P^*; \bar{x}, \bar{y}, \bar{p} \in \mathbb{R}\} \cup \{\bar{x}X^* + \bar{y}Y^* + \bar{q}Q^*; \bar{x}, \bar{y}, \bar{q} \in \mathbb{R}\} \\ &= \bigcup_{\bar{x} \in \mathbb{R}, \varepsilon = \pm 1} \text{Ad}^*G(\bar{x}X^* + \varepsilon P^*) \bigcup_{a^* \in \mathbb{R}, \nu = \pm 1} \text{Ad}^*G(a^*A^* + \nu Q^*) \\ &\quad \bigcup_{a^*, \lambda \in \mathbb{R}} \{a^*A^* + \lambda B^*\}. \end{aligned} \quad (4.1.4)$$

Proof of (4.1.4): Recall that

$$\text{Ad}^*E(\alpha, \alpha, p, q, r, 0) \cdot f = (b^* + r\varepsilon)X^* + (-b^* + pq\varepsilon)Y^* + q\varepsilon P^* - e^{-\alpha}p\varepsilon Q^* + e^{-\alpha}\varepsilon R^*.$$

Let Ψ be a function on $\mathbb{R}\text{-span}\{X^*, Y^*, P^*, Q^*, R^*\}$ defined by

$$\Psi(x, y, p, q, r) := r(y + b^*) + pq \quad \text{for } (x, y, p, q, r) \in \mathbb{R}\text{-span}\{X^*, Y^*, P^*, Q^*, R^*\}.$$

Then $\overline{\Omega_f} \subset \Psi^{-1}(0) \cap (\mathbb{R}^4 \times (\varepsilon\mathbb{R}^{+,0}))$ and $\overline{\Omega_f} \setminus \Omega_f \subset \Psi^{-1}(0) \cap (\mathbb{R}^4 \times \{0\}) = \{(x, y, p, 0, 0); x, y, p \in \mathbb{R}\} \cup \{(x, y, 0, q, 0); x, y, q \in \mathbb{R}\}.$

Conversely, let $\bar{x}, \bar{y}, \bar{p} \in \mathbb{R}$ and suppose $\bar{p} \neq 0$, then

$$\begin{aligned} \text{Ad}^*E(\alpha, \alpha, \frac{\bar{y}+b^*}{q}, q\varepsilon, r\varepsilon, 0)f &= (b^* + r)X^* + \bar{y}Y^* + qP^* - e^{-\alpha}\frac{\varepsilon(\bar{y}+b^*)}{q}Q^* + e^{-\alpha}\varepsilon R^* \\ &\rightarrow \bar{x}X^* + \bar{y}Y^* + \bar{p}P^* \quad \text{as } \alpha \rightarrow \infty, q \rightarrow \bar{p}, r \rightarrow -b^* + \bar{x}. \end{aligned}$$

If $\bar{p} = 0$, we take

$$\begin{aligned} \text{Ad}^*E(\alpha, \alpha, \varepsilon(\bar{y} + b^*)e^{\frac{\alpha}{2}}, e^{-\frac{\alpha}{2}}, r\varepsilon, 0)f &= (b^* + r)X^* + \bar{y}Y^* + e^{-\frac{\alpha}{2}}\varepsilon P^* - (\bar{y} + b^*)e^{-\frac{\alpha}{2}}Q^* + e^{-\alpha}\varepsilon R^* \\ &\rightarrow \bar{x}X^* + \bar{y}Y^* + 0P^* \quad \text{as } \alpha \rightarrow \infty, r \rightarrow -b^* + \bar{x}. \end{aligned}$$

Thus we have $\bar{x}X^* + \bar{y}Y^* + \bar{p}P^* \in \overline{\Omega_f}$. Similarly, we can verify that $\bar{x}X^* + \bar{y}Y^* + \bar{q}Q^* \in \overline{\Omega_f}$. \square

Realization of $\pi_f \in \widehat{G}$: We take the Pukanszky polarization $\mathfrak{q} = \mathbb{R}\text{-span}\{Y, Q, R, S\}$ at f to realize π_f . Noting that $[X, P] = 0$, let

$$E_0(t, u) := \exp(tX + uP) \quad \text{and} \quad h = h(y, q, r, s) := \exp yY \exp(qQ + rR + sS).$$

We realize $\pi_f = \text{ind}_{\exp(\mathfrak{q})}^G \chi_f$ on $L^2(\exp(\mathbb{R}X + \mathbb{R}P))$ as follows: Noting that $[Y, \mathbb{R}X + \mathbb{R}P] \subset \mathbb{R}P$, we write $E_0(t, u)_y := \exp(yY)E_0(t, u)\exp(-yY)$. Let

$\xi \in L^2(\exp(\mathbb{R}X + \mathbb{R}P))$. Then

$$\begin{aligned}
& \pi_f(E_0(x, p)h(y, q, r, s))\xi(E_0(t, u)) \\
&= \chi_f(E_0(t-x, u-p)_{(-y)}^{-1})h(0, q, r, s)E_0(t-x, u-p)_{(-y)}\chi_f(\exp yY) \\
&\quad \left(\frac{\Delta_{\exp(\mathfrak{q})}}{\Delta_G}(\exp(-yY)) \right)^{\frac{1}{2}} \xi(E_0(t-x, u-p)_{-y}) \\
&= \chi_{\text{Ad}^* E_0(t-x, u-p)_{(-y)}(f)}(h(0, q, r, s))\chi_f(\exp yY) \left(\frac{\Delta_{\exp(\mathfrak{q})}}{\Delta_G}(\exp(-yY)) \right)^{\frac{1}{2}} \xi(E_0(t-x, u-p)_{(-y)}) \\
&= \exp \left(-i\varepsilon(e^{-y}q(u-p) - r)e^{x-t} + ib^*y - \frac{y}{2} \right) \xi(t-x, e^{-y}(u-p)).
\end{aligned}$$

We take the left Haar measure dg on G defined by

$$\int_G F(g)dg := \int_{\mathbb{R}^6} F(E_0(x, p)h(y, q, r, s))e^{-y}dx dp dy dq dr ds$$

for integrable functions $F \in L^1(G)$, and define

$$\widehat{F^{\mathfrak{h}}}(g)(l) := \int_{\mathfrak{h}} F(g \exp(qQ + rR + sS))\chi_l(\exp(qQ + rR + sS))dq dr ds,$$

for $l \in \mathfrak{h}^*$ satisfying $l([\mathfrak{h}, \mathfrak{h}]) = \{0\}$. Then

$$\begin{aligned}
& \pi_f(F)\xi(E_0(t, u)) = \int_{\mathbb{R}^6} F(E_0(x, p)h(y, q, r, s))\xi(E_0(t, u))e^{-y}dx dp dy dq dr ds \\
&= \int_{\mathbb{R}^6} F(E_0(x, p)h)\chi_{\text{Ad}^* E_0(t-x, u-p)_{(-y)}(f)}(h(0, q, r, s))\chi_f(\exp yY) \\
&\quad \left(\frac{\Delta_{\exp(\mathfrak{q})}}{\Delta_G}(\exp(-yY)) \right)^{\frac{1}{2}} \xi(E_0(t-x, u-p)_{(-y)})e^{-y}dx dp dy dq dr ds \\
&= \int_{\mathbb{R}^6} F(E_0(t-x, u-p)h)\chi_{\text{Ad}^* E_0(x, p)_{(-y)}(f)}(h(0, q, r, s))\chi_f(\exp yY) \\
&\quad \left(\frac{\Delta_{\exp(\mathfrak{q})}}{\Delta_G}(\exp(-yY)) \right)^{\frac{1}{2}} \xi(E_0(x, p)_{(-y)})e^{-y}dx dp dy dq dr ds \\
&= \int_{\mathbb{R}^6} F(E_0(t-x, u-e^y p)h(y, q, r, s)) \\
&\quad \chi_{\text{Ad}^* E_0(x, p)(f)}(h(0, q, r, s))\chi_f(\exp yY)e^{-\frac{y}{2}}\xi(E_0(x, p))dx dp dy dq dr ds \\
&= \int_{\mathbb{R}^3} \widehat{F^{\mathfrak{h}}}(E_0(t-x, u-e^y p) \exp(yY))(\text{Ad}^* E_0(x, p)(f|_{\mathfrak{h}})e^{ib^*y}e^{-\frac{y}{2}}\xi(E_0(x, p)))dx dp dy \\
&= \int_{\mathbb{R}^3} \widehat{F^{\mathfrak{h}}}(E_0(t-x, u-e^y p) \exp(yY))(-\varepsilon p e^{-x}Q^* + \varepsilon e^{-x}R^*)e^{ib^*y}e^{-\frac{y}{2}}\xi(E_0(x, p))dx dp dy \\
&=: \int_{\mathbb{R}^2} \mathcal{K}_f(t, u; x, p)\xi(E_0(x, p))dx dp,
\end{aligned}$$

where

$$\mathcal{K}_f(t, u; x, p) := \int_{\mathbb{R}} e^{ib^*y}e^{-\frac{y}{2}}\widehat{F^{\mathfrak{h}}}(E(t-x, u-e^y p) \exp yY)(-\varepsilon p e^{-x}Q^* + \varepsilon e^{-x}R^*)d\mathfrak{y} \quad (A.1.5)$$

4.1.4 The orbits of $\nu Q^* + \varepsilon P^*$, $\nu, \varepsilon = \pm 1$

Let $f = \varepsilon P^* + \nu Q^*$ for $\varepsilon, \nu = \pm 1$. Then we have $\mathfrak{g}(f) = \mathbb{R}\text{-span}\{S, R\}$. Let $X^* = \frac{1}{2}(A^* + B^*)$, $Y^* = \frac{1}{2}(A^* - B^*)$. We have

$$\begin{aligned}\Omega_f &= \mathbb{R}A^* + \mathbb{R}B^* + \mathbb{R}^+\varepsilon P^* + \mathbb{R}^+\nu Q^*, \\ \overline{\Omega_f} \setminus \Omega_f &= (\mathbb{R}A^* + \mathbb{R}B^* + \mathbb{R}^+\varepsilon P^*) \cup (\mathbb{R}A^* + \mathbb{R}B^* + \mathbb{R}^+\nu Q^*) \cup (\mathbb{R}A^* + \mathbb{R}B^*) \\ &= \bigcup_{\alpha^* \in \mathbb{R}} (\alpha^* X^* + \mathbb{R}Y^* + \mathbb{R}^+\varepsilon P^*) \bigcup_{a^* \in \mathbb{R}} (a^* A^* + \mathbb{R}B^* + \mathbb{R}^+\nu Q^*) \\ &\quad \bigcup_{a^*, b^* \in \mathbb{R}} \{a^* A^* + b^* B^*\} \\ &= \bigcup_{\alpha^* \in \mathbb{R}} \text{Ad}^*(G)(\alpha^* X^* + \varepsilon P^*) \bigcup_{y^* \in \mathbb{R}} \text{Ad}^*(G)(y^* Y^* + \nu Q^*) \\ &\quad \bigcup_{a^*, b^* \in \mathbb{R}} \{a^* A^* + b^* B^*\}. \tag{4.1.6}\end{aligned}$$

We realize π_f in \widehat{G} by taking a Pukanszky polarization $\mathfrak{l} = \mathbb{R}\text{-span}\{P, Q, R, S\}$ and $\pi_f = \text{ind}_{\exp(\mathfrak{l})}^G \chi_f$. Writing $g_0 = E_0(a, b) := \exp(aA + bB)$, $n = n(p, q, r, s) := \exp(pP) \exp(qQ + rR + sS)$ and $t_0 = E_0(t, u) := \exp(tA + uB)$, we have

$$\begin{aligned}\pi_f(g_0 n) \xi(t_0) &= \chi_f((g_0^{-1} t_0)^{-1} n g_0^{-1} t_0) \xi(g_0^{-1} t_0) = \chi_{\text{Ad}^*(g_0^{-1} t_0) f}(n) \xi(g_0^{-1} t_0) \\ &= \chi_{\text{Ad}^* E_0(t-a, u-b)(f)}(n) \xi(E_0(t-a, u-b)) \\ &= \exp(i(\varepsilon p e^{\frac{-t+a+u-b}{2}} + \nu q e^{-u+b})) \xi(E_0(t-a, u-b)).\end{aligned}$$

Let $F \in L^1(G)$ and $l \in \mathfrak{l}^*$ such that $l[\mathfrak{l}, \mathfrak{l}] = \{0\}$. We define

$$\widehat{F^{\mathfrak{l}}}(g_0)(l) := \int_{\mathfrak{l}} F(g_0 n(p, q, r, s)) e^{il(pP + qQ + rR + sS)} dp dq dr ds.$$

Let $dg_0 dn$ be the left Haar measure transferred the Lebesgue measure $dadb dp dq dr ds$ on \mathfrak{g} by the mapping $(a, b, p, q, r, s) \mapsto g_0(a, b)n(p, q, r, s)$.

$$\begin{aligned}\pi_f(F) \xi(E_0(t, u)) &= \int_G \pi_f(g_0 n) \xi(t_0) F(g_0 n) dg_0 dn \\ &= \int_{\mathbb{R}^6} \chi_{\text{Ad}^* E_0(t-a, u-b)(f)}(n) \xi(E_0(t-a, u-b)) F(E_0(a, b)n) dadb dp dq dr ds \\ &= \int_{\mathbb{R}^6} \chi_{\text{Ad}^* E_0(a, b)(f)}(n) \xi(E_0(a, b)) F(E_0(t-a, u-b)n) dadb dp dq dr ds \\ &=: \int_{\mathfrak{g}/\mathfrak{l}} \widehat{F^{\mathfrak{l}}}(E_0(t-a, u-b)) (\text{Ad}^* E_0(a, b)(f|_{\mathfrak{h}})) \xi(E_0(a, b)) dadb \\ &= \int_{\mathbb{R}^2} \widehat{F^{\mathfrak{l}}}(E_0(t-a, u-b)) (\varepsilon e^{\frac{-a-b}{2}} P^* + \nu e^{-b} Q^*) \xi(E_0(a, b)) dadb \\ &=: \int_{\mathbb{R}^2} \mathcal{K}_F(t, u; a, b) \xi(E_0(a, b)) dadb,\end{aligned}$$

where

$$\begin{aligned}\mathcal{K}_F(t, u; a, b) &:= \widehat{F^{\mathfrak{l}}}(E_0(t-a, u-b)) (\text{Ad}^* E_0(a, b)(f|_{\mathfrak{l}})) \\ &= \widehat{F^{\mathfrak{l}}}(E_0(t-a, u-b)) (\varepsilon e^{\frac{-a-b}{2}} P^* + \nu e^{-b} Q^*). \tag{4.1.7}\end{aligned}$$

4.1.5 The orbits of $a^*A^* + \varepsilon Q^*$, $a^* \in \mathbb{R}, \varepsilon = \pm 1$

Let $f := a^*A^* + \varepsilon Q^*$. Then $\mathfrak{g}(a^*A^* + \varepsilon Q^*) = \mathbb{R}\text{-span}\{A, P, R, S\}$,

$$\Omega_f = a^*A^* + \mathbb{R}B^* + \mathbb{R}^+\varepsilon Q^*, \text{ and}$$

$$\overline{\Omega_f} \setminus \Omega_f = a^*A^* + \mathbb{R}B^* + 0 \cdot Q^* = \cup_{\lambda \in \mathbb{R}} (a^*A^* + \lambda B^*). \quad (4.1.8)$$

We realize π_f in \widehat{G} as $\pi_f = \text{ind}_{\exp(\mathfrak{l}_A)}^G \chi_f$, $\mathfrak{l}_A = \mathbb{R}\text{-span}\{A, P, Q, R, S\}$, on $L^2(G/\exp(\mathfrak{l}_A), \chi_f) \simeq L^2(\exp \mathbb{R}B)$. Write

$$E_0(b) := \exp bB, \quad h = h(a, p, q, r, s) := \exp(aA) \exp(pP) \exp(qQ + rR + sS)$$

for $b \in \mathbb{R}$, $(a, p, q, r, s) \in \mathbb{R}^5$. We have

$$\begin{aligned} \pi_f(E_0(b)h(a, p, q, r, s))\xi(E_0(t)) &= \chi_{\text{Ad}^*E_0(t-b)(f)}(h)\xi(E_0(t-b)) \\ &= \exp(i(aa^* + q\varepsilon e^{b-t}))\xi(E_0(t-b)), \end{aligned}$$

$$\begin{aligned} \pi_f(F)\xi(E_0(t)) &= \int_{\mathbb{R}^6} F(E_0(b)h)\chi_{\text{Ad}^*E_0(t-b)(f)}(h)\xi(E_0(t-b))dbdadpdqdrds \\ &= \int_{\mathbb{R}^6} F(E_0(t-b)h)\chi_{\text{Ad}^*E_0(b)(f)}(h)\xi(E_0(b))dbdadpdqdrds \\ &= \int_{\mathbb{R}} \widehat{F^{\mathfrak{l}_A}}(E_0(t-b))(\text{Ad}^*E_0(b)(f|_{\mathfrak{l}_A}))\xi(E_0(b))db \\ &= \int_{\mathbb{R}} \mathcal{K}_F(t, b)\xi(E_0(b))db, \end{aligned}$$

where

$$\widehat{F^{\mathfrak{l}_A}}(g)(l) := \int_{\mathfrak{l}_A} F(gh(a, p, q, r, s))\chi_l(h(a, p, q, r, s))dadpdqdrds$$

for $l \in \mathfrak{g}^*$ such that $l([\mathfrak{l}_A, \mathfrak{l}_A]) = \{0\}$, and

$$\mathcal{K}_F(t, b) := \widehat{F^{\mathfrak{l}_A}}(E_0(t-b))(\text{Ad}^*E_0(b)(f|_{\mathfrak{l}_A})) = \widehat{F^{\mathfrak{l}_A}}(E_0(t-b))(a^*A^* + \varepsilon e^{-b}Q^*).$$

4.1.6 The orbits of $x^*X^* + \varepsilon P^*$, $x^* \in \mathbb{R}, \varepsilon = \pm 1$

Let $f = \alpha^*(\frac{A^*+B^*}{2}) + \varepsilon P^* = \alpha^*X^* + \varepsilon P^*$, $X = A+B$, $Y = A-B$, $X^* = \frac{A^*+B^*}{2}$, $Y^* = \frac{A^*-B^*}{2}$. Then $\mathfrak{g}(f) = \mathbb{R}\text{-span}\{X, Q, R, S\}$. By (3.1.11) and (3.1.12), we have

$$\Omega_f = \alpha^*X^* + \mathbb{R}Y^* + \mathbb{R}^+\varepsilon P^*,$$

$$\overline{\Omega_f} \setminus \Omega_f = \alpha^*X^* + \mathbb{R}Y^* = \bigcup_{\lambda \in \mathbb{R}} \{\alpha^*X^* + \lambda Y^*\}. \quad (4.1.9)$$

We realize π_f in \widehat{G} as $\pi_f = \text{ind}_{\exp(\mathfrak{l}_X)}^G \chi_f$, where $\mathfrak{l}_X = \mathbb{R}\text{-span}\{X, P, Q, R, S\}$, on $L^2(G/\exp(\mathfrak{l}_X), \chi_f) \simeq L^2(\exp \mathbb{R}Y)$. We write

$$E_0(y) := \exp(yY), \quad h = h(x, p, q, r, s) := \exp(xX) \exp(pP) \exp(qQ + rR + sS)$$

for $y \in \mathbb{R}$ and $(x, p, q, r, s) \in \mathbb{R}^5$, and transfer a Lebesgue measure on \mathbb{R}^6 to a left Haar measure on G by $(x, y, p, q, r, s) \mapsto E_0(y)h(x, p, q, r, s)$. For $\xi \in L^2(G/\exp(\mathfrak{l}_X), \chi_f)$ we have

$$\begin{aligned} \pi_f(E_0(y)h(x, p, q, r, s))\xi(E_0(t)) &= \chi_{\text{Ad}^* E_0(t-y)(f)}(h)\xi(E_0(t-y)) \\ &= \exp(ix\alpha^* + ip\epsilon e^{-t+y})\xi(E_0(t-y)), \end{aligned}$$

and for an integrable function F , we have

$$\begin{aligned} \pi_f(F)\xi(E_0(t)) &= \int_{\mathbb{R}^6} F(E_0(y)h(x, p, q, r, s))\pi_f(E_0(y)h(x, p, q, r, s))\xi(E_0(t))dydx dpdqdrds \\ &= \int_{\mathbb{R}^6} F(E_0(y)h)\chi_{\text{Ad}^* E_0(t-y)(f)}(h)\xi(E_0(t-y))dydx dpdqdrds \\ &= \int_{\mathbb{R}^6} F(E_0(t-y)h)\chi_{\text{Ad}^* E_0(y)(f)}(h)\xi(E_0(y))dydx dpdqdrds \\ &=: \int_{\mathbb{R}} \widehat{F^{\mathfrak{l}_X}}(E_0(t-y))(\text{Ad}^* E_0(y)(f|_{\mathfrak{h}}))\xi(E_0(y))dy \\ &= \int_{\mathbb{R}} \widehat{F^{\mathfrak{l}_X}}(E_0(t-y))(\alpha^* X^* + \epsilon e^{-y} P^*)\xi(E_0(y))dy. \end{aligned}$$

For $l \in \mathfrak{g}^*$ satisfying $l([\mathfrak{l}_X, \mathfrak{l}_X]) = \{0\}$, we write

$$\widehat{F^{\mathfrak{l}_X}}(g)(l) := \int_{\mathfrak{l}_X} e^{il(Z)} F(g \exp Z) dZ, \quad g \in G,$$

with some fixed Lebesgue measure dZ on \mathfrak{h} . And $\pi_f(F)$ is described by the operator $\pi_f(F)\xi(E_0(y)) = \int_{\mathbb{R}} \mathcal{K}_f(t, y)\xi(E_0(y))dy$, where

$$\mathcal{K}_f(t, y) := \widehat{F^{\mathfrak{l}_X}}(E_0(t-y))(\text{Ad}^* E_0(y)(f|_{\mathfrak{l}_X})) = \widehat{F^{\mathfrak{l}_X}}(E_0(t-y))(\alpha^* X^* + \epsilon e^{-y} P^*). \quad (4.1.10)$$

5 The continuity and infinity conditions

In this short section we find the continuity and infinity conditions for $C^*(G)$.

Definition 5.1. We say that a net $(\gamma_i)_{i \in I}$ in a topological space Γ goes to infinity, if the net contains no converging subnet. In particular, a sequence of orbits $\overline{\Omega} = (\Omega_k)_{k \in \mathbb{N}} \subset \mathfrak{g}^*$ goes to infinity, if for any compact subset $K \subset \mathfrak{g}^*$ there exists an index k_0 such that $K \cap \Omega_k = \emptyset$ whenever $k \geq k_0$.

Proposition 5.2 (Riemann-Lebesgue Lemma). *Let A be a C^* -algebra. If a net $(\pi_k)_k \subset \widehat{A}$ goes to infinity, then $\lim_k \|\pi_k(a)\|_{\text{op}} = 0$ for all $a \in A$.*

Proof. We know from [Di, Proposition 3.3.7] that for every $c > 0$ and $a \in A$, the subset $\{\pi \in \widehat{A}; \|\pi(a)\|_{\text{op}} \geq c\}$ is quasi-compact. This shows that $\lim_k \|\pi_k(a)\|_{\text{op}} = 0$, if the net $(\pi_k)_k$ goes to infinity. \square

In the following we recall the definition of properly converging sequences and their limit sets, we also give a proposition of properly converging sequences in our group.

Definition 5.3. Let Y be a topological space. Let $\bar{y} = (y_k)_k$ be a net in Y . We denote by $L(\bar{y})$ the set of all limit points of the net \bar{y} . A net \bar{y} is called *properly converging* if \bar{y} has limit points and if every cluster point of the net is a limit point, i.e. the set of limit points of any subnet is always the same, indeed, it equals to $L(\bar{y})$.

We know that every converging net in Y admits a properly converging subnet, hence, we can work with properly converging nets in our space \widehat{G} .

Definition 5.4. Let $\varepsilon = \pm 1$, and

- $\mathbb{R}_{\varepsilon P^*} := \{ \text{Ad}^*(G)(x^*X^* + \varepsilon P^*) = x^*X^* + \text{Ad}^*(G)(\varepsilon P^*), x^* \in \mathbb{R} \},$
- $\mathbb{R}_{\varepsilon Q^*} := \{ \text{Ad}^*(G)(a^*A^* + \varepsilon Q^*) = a^*A^* + \text{Ad}^*(G)(\varepsilon Q^*), a^* \in \mathbb{R} \},$
- $\mathbb{R}_{\varepsilon R^*} := \{ \text{Ad}^*(G)(b^*B^* + \varepsilon R^*) = b^*B^* + \text{Ad}^*(G)(\varepsilon R^*), b^* \in \mathbb{R} \},$
- $\mathbb{R}_{\varepsilon S^*} := \{ \text{Ad}^*(G)(b^*B^* + \varepsilon S^*) = b^*B^* + \text{Ad}^*(G)(\varepsilon S^*), b^* \in \mathbb{R} \}.$

Proposition 5.5. Let $\underline{\Omega} = (\Omega_k)_{k \in \mathbb{N}}$ be a properly converging sequence in \mathfrak{g}^*/G . Then there exists a subsequence (also denoted by $\underline{\Omega} = (\Omega_k)_{k \in \mathbb{N}}$ for simplicity) such that either $\underline{\Omega}$ is a constant sequence, or if $\underline{\Omega}$ is not constant, it is contained in one of the subsets in Definition 5.4 and its limit set is the closure in \mathfrak{g}^*/G of one of the points of this subset or all the elements of the subsequence are characters and the limit set is a single character.

Proof. The orbit space consists of four open orbits of the eight sets given in Definition 5.4, of the four orbits of $\nu P^* + \varepsilon Q^*$, $\varepsilon, \nu \in \{1, -1\}$, and of the set of characters \mathfrak{X} . Hence, if the sequence $\underline{\Omega}$ contains a constant subsequence, then the limit set of the sequence itself is the closure of this constant element, since it is properly converging. If the sequence is not constant, then we can suppose that a subsequence is contained in one of the 8 sets or is made out of characters. If we deal with non-characters, then we can write $\underline{\Omega} = (\Omega + \lambda_k U^*)_{k \in \mathbb{N}}$, where $\lambda_k \in \mathbb{R}$ and U^* is one of the characters X^*, A^* or B^* , and Ω is a fixed orbit in one of sets $\mathbb{R}_{\varepsilon P^*}, \mathbb{R}_{\varepsilon Q^*}, \mathbb{R}_{\varepsilon R^*}, \mathbb{R}_{\varepsilon S^*}$. Since the sequence $\underline{\Omega}$ is properly converging, for any v in a limit orbit Ω_∞ , there exists a sequence $(v_k)_k \subset \Omega$ and a real sequence $(\lambda_k)_k$ such that $\lambda_k U^* + v_k$ converges in \mathfrak{g}^* to v . We can write $\mathfrak{g} = \mathbb{R}U + \mathfrak{m}$, where \mathfrak{m} is an ideal of \mathfrak{g} containing $[\mathfrak{g}, \mathfrak{g}]$. Then by the formulas in Subsection 3.1, it follows that $v_k(U) = \varphi(v_k|_{\mathfrak{m}})$, where $\varphi : \mathfrak{m}^* \rightarrow \mathbb{R}$ is a continuous function. Hence the sequence $(\lambda_k + \varphi(v_k|_{\mathfrak{m}}))_k$ is convergent with a limit $\lambda + v(U)$ and so the sequence $(\lambda_k)_k$ converges to some λ . Hence the limit orbit Ω_∞ is contained in $\overline{\Omega + \lambda U^*} = \lambda U^* + \overline{\Omega}$, and so the limit set of the sequence $\underline{\Omega}$ is the set $\overline{\Omega + \lambda U^*}$. \square

We define now the subsets $S_j, j = 0, \dots, 6$, which we shall need for the definition of the algebra $D^*(G)$ in the later section.

Definition 5.6. Let

- $\Gamma_0 = \mathfrak{X} = \{a^* A^* + b^* B^*, a^*, b^* \in \mathbb{R}\},$
- $\Gamma_1 := \{\mathbb{R}_{\varepsilon P^*}, \varepsilon = \pm 1\},$
- $\Gamma_2 := \{\mathbb{R}_{\nu Q^*}, \nu = \pm 1\},$
- $\Gamma_3 := \{\Omega_{\varepsilon P^* + \nu Q^*}, \nu, \varepsilon = \pm 1\},$
- $\Gamma_4 := \{\mathbb{R}_{\varepsilon R^*}, \varepsilon = \pm 1\},$
- $\Gamma_5 := \{\mathbb{R}_{\varepsilon S^*}, \varepsilon = \pm 1\},$
- $\Gamma_6 := \{\Omega_{\varepsilon S^* + \nu Q^*}, \varepsilon, \nu = \pm 1\}.$

Let $S_i := \bigcup_{j=0}^i \Gamma_j \subset \mathfrak{g}^*/G$. Then S_i is closed in S_{i+1} for $i = 0, \dots, 5$ with respect to the orbit topology. The set Γ_6 is finite and open in \widehat{G} while Γ_3 is finite and open in S_3 . The sets $\Gamma_1, \Gamma_2, \Gamma_4$ and Γ_5 are homeomorphic to two disjoint copies of the real line \mathbb{R} .

Theorem 5.7. *The mappings $\pi \mapsto \pi(F)$, $F \in C^*(G)$, are norm-continuous on the subsets Γ_i of \widehat{G} for all $i = 0, \dots, 6$.*

Proof. This follows from the following more general result for the C^* -algebra of a second countable locally compact group G .

Let $(\chi_k)_k$ be a sequence of unitary characters of G , which converges pointwise to a unitary character χ . Let (π, \mathcal{H}) be a unitary representation of G . Then for every $F \in L^1(G)$, we have that

$$\lim_{k \rightarrow \infty} \|\chi_k \otimes \pi(F) - \chi \otimes \pi(F)\|_{\text{op}} = 0.$$

Indeed, it suffices to remark that

$$\|\chi_k \otimes \pi(F) - \chi \otimes \pi(F)\|_{\text{op}} = \|\pi(\chi_k \cdot F) - \pi(\chi \cdot F)\|_{\text{op}} \leq \|\chi_k \cdot F - \chi \cdot F\|_1$$

and therefore by the Lebesgue theorem of dominated convergence, $\lim_{k \rightarrow \infty} \|\chi_k \otimes \pi(F) - \chi \otimes \pi(F)\|_{\text{op}} = 0$. \square

6 The boundary conditions for the C^* -algebra

For each one of our subsets Γ_i defined in Definition 5.6, we must find the linear mappings $\sigma_{i,\delta}, \delta > 0$, which give the structure of the C^* -algebra of G an almost $C_0(\mathcal{K})$ -algebra. These mappings $\sigma_{i,\delta}$ will be built on the representations coming from the different sets Γ_i . Hence, this forces us to make a case by case study of the different situations.

We indicate here some definitions and methods, which will be used in the proofs of this (long) section.

- Remark 6.1.** 1. For a measurable subset S of a measure space (X, μ) , denote the multiplication operator with the characteristic function 1_S on $L^2(X, \mu)$ by M_S .
2. Let $H = \exp(\mathfrak{h})$ be an abelian normal subgroup of G . The subspace $L_c^1 = L_{c, \mathfrak{h}}^1$ is defined to be the set of all F 's in $L^1(G)$, for which the partial Fourier transform

$$\widehat{F}^{\mathfrak{h}}(s, q) := \int_H F(sh) \chi_q(h) dh, s \in G, q \in \mathfrak{h}^*$$

is a C^∞ -function with compact support on $G/H \times \mathfrak{h}^*$. The vector space L_c^1 is of course dense in $L^1(G)$ and hence is also dense in $C^*(G)$.

Furthermore, for every $F \in L_c^1$ there exists a function $\varphi \in C_c(G/H)$ such that

$$|\widehat{F}^{\mathfrak{h}}(s, q)| \leq \|q\| |\varphi(s)| \text{ for } s \in G, q \in \mathfrak{h}^*.$$

3. Let $F : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$ be a smooth function with compact support for which there exists some continuous function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^+$ with compact support, such that

$$|F(x, y)| \leq \varphi(x - y) \text{ for } x, y \in \mathbb{R}^d.$$

Then by the Young's inequality, the operator norm $\|\cdot\|_{\text{op}}$ of the kernel operator T_F defined on $L^2(\mathbb{R}^d)$ by

$$T_F(\xi)(x) := \int_{\mathbb{R}^d} F(x, y) \xi(y) dy, \xi \in L^2(\mathbb{R}^d) \text{ and } x \in \mathbb{R}^d,$$

is bounded by the L^1 -norm $\|\varphi\|_1$ of φ .

4. Let (X, μ) and (Y, ν) be two measure spaces. Let $Y \mapsto L^2(X, \mu) : y \mapsto \xi(y)$ be an integrable mapping. Then we have that

$$\left\| \int_Y \xi(y) d\nu(y) \right\|_2 \leq \int_Y \|\xi(y)\|_2 d\nu(y),$$

this is,

$$\left(\int_X \left| \int_Y \xi(y)(x) d\nu(y) \right|^2 d\mu(x) \right)^{1/2} \leq \int_Y \left(\int_X |\xi(y)(x)|^2 d\mu(x) \right)^{1/2} d\nu(y).$$

Proposition 6.2. Let (X, μ) be a measure space, and $(\sigma_i)_{i \in I}$ be a family of bounded linear operators on the Hilbert space $\mathcal{H} = L^2(X, \mu)$ such that $\|\sigma_i\|_{\text{op}} \leq C$ for all $i \in I$ and some $C > 0$. Suppose furthermore that there exist families $(T_{i,j})_{i \in I} (j = 1, \dots, N)$ and $(S_i)_{i \in I}$ of measurable subsets of X such that $T_{i,j} \cap$

$T_{i',j} = \emptyset, j = 1, \dots, N$, and $S_i \cap S_{i'} = \emptyset$ whenever $i \neq i'$. Then the linear operator

$$\sigma = \sum_{j=1}^N \sum_{i \in I} M_{T_{i,j}} \circ \sigma_i \circ M_{S_i}$$

is bounded by NC .

Proof. Let us write

$$\sigma^j := \sum_{i \in I} M_{T_{i,j}} \circ \sigma_i \circ M_{S_i} \quad \text{for } j = 1, \dots, N.$$

Then $\sigma = \sum_{j=1}^N \sigma^j$ and for $\xi \in L^2(X, \mu), j \in \{1, \dots, N\}$ we have

$$\begin{aligned} \|\sigma^j(\xi)\|_2^2 &= \int_X \left| \sum_{i \in I} M_{T_{i,j}} \circ \sigma_i \circ M_{S_i}(\xi)(x) \right|^2 d\mu(x) \\ &= \sum_{i \in I} \int_{T_{i,j}} |\sigma_i(M_{S_i}(\xi))(x)|^2 d\mu(x) \\ &\leq \sum_{i \in I} \int_X |\sigma_i(M_{S_i}(\xi))(x)|^2 d\mu(x) \\ &\leq \sum_{i \in I} C^2 \int_X |M_{S_i}(\xi)(x)|^2 d\mu(x) \\ &= C^2 \left(\sum_{i \in I} \int_{S_i} |\xi(x)|^2 d\mu(x) \right) \\ &\leq C^2 \int_X |\xi(x)|^2 d\mu(x) \\ &= C^2 \|\xi\|_2^2. \end{aligned}$$

Hence $\|\sigma\|_{\text{op}} \leq NC$. □

6.1 The open orbits

6.1.1 The orbit $\Omega_{\varepsilon, -\varepsilon} = \Omega_{\varepsilon S^* - \varepsilon Q^*}$ for $\varepsilon = \pm 1$

Let $\ell_{\varepsilon, -\varepsilon} := \varepsilon(S^* - Q^*), \varepsilon \in \{1, -1\}$. We have seen in (3.1.2) and (3.1.3) that the G -orbit of $(\ell_{\varepsilon, -\varepsilon})|_{\mathfrak{h}}$ is the subset

$$\Omega_{\varepsilon, -\varepsilon}|_{\mathfrak{h}} = \left\{ \varepsilon \left((-e^{-b} + \frac{e^{-b}p^2}{2})Q^* + (-e^{-\frac{a+b}{2}}p)R^* + e^{-a}S^* \right), a, b, p \in \mathbb{R} \right\}.$$

By Subsection 4.1.1, the boundary $\partial(\varepsilon, -\varepsilon)$ of the orbit of $(\ell_{\varepsilon, -\varepsilon})|_{\mathfrak{h}}$ is given by the union of the 5 orbits

$$\begin{aligned} G \cdot \varepsilon S|_{\mathfrak{h}}^* &= \{E(a, b, p) \cdot \varepsilon S^* = (e^{-b} \frac{\varepsilon}{2} p^2) Q^* + (-\varepsilon p e^{-\frac{a+b}{2}}) R^* + \varepsilon e^{-a} S^*\}; \ a, b, p \in \mathbb{R}\}, \\ G \cdot \varepsilon Q|_{\mathfrak{h}}^* &= \{E(a, b, p) \cdot \varepsilon Q^* = \varepsilon e^{-b} Q^*\}; \ a, b, p \in \mathbb{R}\}, \\ G \cdot (-\varepsilon Q^*)|_{\mathfrak{h}} &= \{-E(a, b, p) \cdot \varepsilon Q^* = -\varepsilon e^{-b} Q^*\}; \ a, b, p \in \mathbb{R}\}, \\ G \cdot \sqrt{2} \varepsilon R|_{\mathfrak{h}}^* &= \{E(a, b, p) \cdot (\sqrt{2} \varepsilon R^*) = \varepsilon((-e^{-b} \sqrt{2} p) Q^* + (e^{-\frac{a-b}{2}} \sqrt{2}) R^*)\}; \ a, b, p \in \mathbb{R}\}, \\ G \cdot (-\sqrt{2} \varepsilon R^*)|_{\mathfrak{h}} &= \{-E(a, b, p) \cdot (\sqrt{2} \varepsilon R^*) = \varepsilon((e^{-b} \sqrt{2} p) Q^* - (e^{-\frac{a-b}{2}} \sqrt{2}) R^*)\}; \ a, b, p \in \mathbb{R}\} \end{aligned}$$

and $\{0\}$. Here we denote $E(a, b, c) = \exp(aA) \exp(bB) \exp(pP)$ for $(a, b, p) \in \mathbb{R}$.

We must now consider the following subsets of \mathbb{R}^3 , which give us the points in the orbit $\Omega_{\varepsilon, -\varepsilon}|_{\mathfrak{h}}$ “close” up to a distance δ to the corresponding boundary orbits.

Definition 6.3. 1. For $k \in \mathbb{Z}$, denote by

$$J_{\delta, k} := \left[\log\left(\frac{1}{\delta^6}\right), +\infty \right) \times \left(-\infty, \log\left(\frac{1}{\delta}\right) \right] \times I_{\delta^2, k},$$

where $I_{c, l} := [cl, cl + c) \subset \mathbb{R}$ for $l \in \mathbb{Z}, c > 0$ and let

$$L_{\delta, k} := \mathbb{R} \times \mathbb{R} \times I_{\delta^2, k}.$$

2. Let

$$K_{\delta} = \left[-\frac{1}{\delta}, \frac{1}{\delta}\right] \times \left[-\frac{1}{\delta}, \frac{1}{\delta}\right] \times \left[-\frac{1}{\delta^{1/2}}, \frac{1}{\delta^{1/2}}\right].$$

3. Let $S_{\delta, 1} := \{(a, b, p); e^{-a} > \delta^6\}$ (corresponds to $\mathbb{R}_{\varepsilon S^*}$).

4. Let $S_{\delta, 2} := \{(a, b, p); e^{-a} \leq \delta^6, e^{-b} < \delta\}$ (corresponds to $\mathbb{R}_{\varepsilon S^*}$).

5. Let

$$S_{\delta, 3, k, +} := J_{\delta, k} \cap \{(a, b, p); \varepsilon e^{-b/2} (1 - \frac{p^2}{2}) > \delta^{1/2}, \delta \leq e^{-b} \leq \frac{1}{\delta^{5/4}}\} \text{ for } k \in \mathbb{Z},$$

(corresponds to $\mathbb{R}_{\varepsilon Q^*}$) and $S_{\delta, 3, +} := \bigcup_k S_{\delta, 3, k, +}$.

6. Let

$$S_{\delta, 3, k, -} := J_{\delta, k} \cap \{(a, b, p); \varepsilon e^{-b/2} (1 - \frac{p^2}{2}) < -\delta^{1/2}, \delta \leq e^{-b} \leq \frac{1}{\delta^{5/4}}\} \text{ for } k \in \mathbb{Z},$$

(corresponds to $\mathbb{R}_{-\varepsilon Q^*}$) and $S_{\delta, 3, -} := \bigcup_k S_{\delta, 3, k, -}$.

7. Let

$$S_{\delta, 4, \pm} := \{(a, b, p) \in \mathbb{R}^3; e^{-a} < \delta^6, e^{-b} \geq \delta, e^{-b/2} |\frac{p^2}{2} - 1| \leq \delta^{1/2}, \pm p \leq 0\}$$

(corresponds to $\mathbb{R}_{\pm R^*}$).

8. Define the corresponding multiplication operators on $L^2(\mathbb{R}^3)$ by $M_{\delta,i} = M_{S_{\delta,i}}$ for $i = 1, 2$, respectively; for $k \in \mathbb{Z}$, define $M_{\delta,3,k,\pm} = M_{S_{\delta,3,k,\pm}}$, $M_{\delta,3,\pm} = M_{S_{\delta,3,\pm}}$ and $M_{\delta,4,\pm} = M_{S_{\delta,4,\pm}}$.

Remark 6.4.

1. For any $\delta > 0$, the sets $\{S_{\delta,1}, S_{\delta,2}, S_{\delta,3,k,\pm}, S_{\delta,4,\pm}; k \in \mathbb{Z}\}$ are pairwise disjoint measurable subsets of \mathbb{R}^3 .
2. The region

$$\{(a, b, p); e^{-b/2}|\frac{p^2}{2} - 1| > \delta^{1/2}, e^{-b} > \frac{1}{\delta^{5/4}}\}$$

does not need to be considered, since for a function $F \in L_c^1$ the kernel function $\widehat{F}^{\mathfrak{h}}(s - a, t - b, e^{\frac{a-b}{2}}(v - p), \varepsilon(-e^{-b} + \frac{e^{-b}p^2}{2})Q^* - (e^{-\frac{a+b}{2}}p)R^* + e^{-a}S^*)$ of the operator $\pi_{\varepsilon, -\varepsilon}(F)$ is 0 for δ small enough on this region.

3. Recall that for $\alpha = (a, b, p)$ and $\beta = (s, t, v) \in \mathbb{R}^3$, the multiplication of $E(s, t, v) \cdot E(a, b, p)$ is given by

$$E(s, t, v) \cdot E(a, b, p) = E(s + a, t + b, e^{\frac{-a+b}{2}}v + p).$$

For $\beta \in K_\delta$ and $\alpha \in S_{\delta,3,k,\pm}$, we have that $e^{-a} \leq \delta^6, e^b < \frac{1}{\delta}, |v| \leq \frac{1}{\delta^{1/2}}$ and so $e^{\frac{-a+b}{2}}|v| < \delta^2$ and

$$\begin{aligned} \beta \cdot \alpha &= (s + a, t + b, e^{\frac{-a+b}{2}}v + p) \\ &\in \mathbb{R} \times \mathbb{R} \times ([-\delta^2, \delta^2] + I_{\delta^2,k}) \\ &\subset \bigcup_{i=-1}^1 L_{\delta,k+i} =: T_{\delta,3,k}. \end{aligned} \tag{6.1.1}$$

We see that the set $T_{\delta,3,k}$ is the union of the 3 disjoint boxes $L_{\delta,k+i}$ for $i \in \{-1, 0, 1\}$. Let $N_{\delta,3,k}$ be the multiplication operator with the characteristic function of the set $T_{\delta,3,k}$ for all $k \in \mathbb{Z}$.

4. The irreducible representation $\pi_{\varepsilon, -\varepsilon} = \text{ind}_H^G \chi_{\varepsilon(S^* - Q^*)}$ acts on $\xi \in L^2(G/H, \chi_{\varepsilon(S^* - Q^*)})$ in the following way (see (4.1.1)):

$$\begin{aligned} &\pi_{\varepsilon, -\varepsilon}(F)\xi(E(s, t, v)) \\ &= \int_{\mathbb{R}^3} e^{\frac{1}{2}(a-b)} \widehat{F}^{\mathfrak{h}} \left(s - a, t - b, e^{\frac{a-b}{2}}(v - p), \varepsilon(-e^{-b}(1 - \frac{p^2}{2})Q^* + (-e^{-\frac{a+b}{2}}p)R^* + e^{-a}S^*) \right) \\ &\quad \xi(E(a, b, p)) da db dp \quad \text{for } F \in L^1(G). \end{aligned}$$

5. Let $F \in L_c^1$. There exists a box $K = [-M, M]^3$ for some $M > 0$ such that

$$\widehat{F}^{\mathfrak{h}}(s, q) = 0 \text{ for } s \notin K \text{ and } q \in \mathfrak{h}^*.$$

Hence it follows from the relations (6.1.1) that for $\delta > 0$ small enough,

$$\widehat{F}^h(st^{-1}, t \cdot (\varepsilon(S^* - Q^*))) = 0 \text{ for } t \in S_{\delta,3,k,\pm}, s \notin T_{\delta,3,k} \text{ and } k \in \mathbb{Z}.$$

Therefore,

$$N_{\delta,3,k} \circ \pi_{\varepsilon,-\varepsilon}(F) \circ M_{\delta,3,k,\pm} = \pi_{\varepsilon,-\varepsilon}(F) \circ M_{\delta,3,k,\pm} \text{ for } k \in \mathbb{Z}. \quad (6.1.2)$$

Definition 6.5. Let $\sigma_l := \text{ind}_H^G \chi_l$ for $l = \varepsilon S^*$, $\varepsilon(\pm\sqrt{2}R^* - 2Q^*)$ and $\varepsilon(-1 + \frac{\delta^4 k^2}{2})Q^*$, respectively. We define the linear operator $\sigma_{\varepsilon,-\varepsilon,\delta}(F)$ by the rule

$$\sigma_{\varepsilon,-\varepsilon,\delta}(F) := \sum_{i=1}^2 \sigma_{\varepsilon S^*}(F) \circ M_{\tilde{\varphi}_i} \circ M_{\delta,i} + \sigma_{\delta,3}(F) + \sigma_{\delta,4,+}(F) + \sigma_{\delta,4,-}(F),$$

where $\sigma_{\delta,4,\pm}(F) = \sigma_{\varepsilon(\pm\sqrt{2}R^* - 2Q^*)}(F) \circ M_{\delta,4,\pm}$ and

$$\sigma_{\delta,3}(F) := \sum_{k \in \mathbb{Z}} \left(N_{\delta,3,k} \circ \sigma_{\varepsilon(-1 + \frac{\delta^4 k^2}{2})Q^*}(F) \circ (M_{\delta,3,k,+} + M_{\delta,3,k,-}) \right),$$

where the operators $M_{\tilde{\varphi}_i}$, $i = 1, 2$, are multiplication operators with the functions $\tilde{\varphi}_1(a, b, p) := \varphi(-e^{-b})$, $\tilde{\varphi}_2 = 1$ and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is a C^∞ -function with support contained in $[-1, 1]$ such that $\varphi([-1, 1]) = \{1\}$ and $\|\varphi\|_\infty = 1$.

The kernel functions of the different operators appearing in the sum above are given by the following.

1. $\sigma_{\varepsilon S^*}(F) \circ M_{\tilde{\varphi}_i}$ gives

$$F_{\delta,i}((s, t, v), (a, b, p)) = \widehat{F}^h(s - a, t - b, e^{\frac{a-b}{2}}(v - p), \varepsilon((\frac{e^{-b}p^2}{2})Q^* + (-e^{-\frac{a+b}{2}}p)R^* + e^{-a}S^*)) \\ e^{\frac{1}{2}(a-b)}\varphi_i(a, b, p) \text{ for } i = 1, 2;$$

2. $\sigma_{\varepsilon(\pm\sqrt{2}R^* - 2Q^*)}(F) \circ M_{\delta,4,\pm}$ gives

$$F_{\delta,4,\pm}((s, t, v), (a, b, p)) = \widehat{F}^h(s - a, t - b, e^{\frac{a-b}{2}}(v - p), \varepsilon(-e^{-b}(\pm\sqrt{2}p + 2))Q^* \pm \varepsilon(e^{-\frac{a+b}{2}}\sqrt{2})R^*) \\ e^{\frac{1}{2}(a-b)}1_{S_{\delta,4,\pm}}(a, b, p); \text{ and}$$

3. $\sigma_{\varepsilon(-1 + \frac{\delta^4 k^2}{2})Q^*}(F) \circ M_{\delta,3,k,\pm}$ gives

$$F_{\delta,3,k,\pm}((s, t, v), (a, b, p)) = \widehat{F}^h(s - a, t - b, e^{\frac{a-b}{2}}(v - p), \varepsilon(e^{-b}(-1 + \frac{\delta^4 k^2}{2})Q^*)) \\ e^{\frac{1}{2}(a-b)}1_{S_{\delta,3,k,\pm}}(a, b, p) \text{ for } k \in \mathbb{Z}.$$

Now the representation $\sigma_{\varepsilon|1 - \frac{k^2\delta^4}{2}|Q^*}$ is equivalent to the representation $\sigma_{\varepsilon Q^*}$, since both linear functionals are in the same G -orbit. Therefore, for any $k \in \mathbb{Z}$ and $1 > \delta > 0$:

$$\|N_{\delta,3,k} \circ \sigma_{\varepsilon|1 - \frac{\delta^4 k^2}{2}|Q^*}(F) \circ M_{\delta,3,k,\pm}\|_{\text{op}} \leq \|\sigma_{\varepsilon Q^*}(F)\|_{\text{op}}. \quad (6.1.3)$$

Proposition 6.6. *For any $F \in L_c^1$ and $\delta > 0$, the operator $\sigma_{\delta,3}(F)$ is bounded in norm by $C\|\sigma_{Q^*}(F)\|_{\text{op}}$, where C is the number given by $\max_{l \in \mathbb{Z}} \#\{k \in \mathbb{Z}; T_{\delta,3,k} \cap T_{\delta,3,l} \neq \emptyset\}$ which is less and equal 9.*

Proof. Apply Proposition 6.2. □

Proposition 6.7. *For any $F \in L_c^1$ and $\delta > 0$, we have that*

$$\|\sigma_{\varepsilon,-\varepsilon,\delta}(F)\|_{\text{op}} \leq 2\|\sigma_{\varepsilon S^*}(F)\|_{\text{op}} + C\|\sigma_{\varepsilon Q^*}(F)\|_{\text{op}} + 2\|\sigma_{\varepsilon R^*}(F)\|_{\text{op}}$$

and we can extend the mapping $F \mapsto \sigma_{\varepsilon,-\varepsilon,\delta}(F)$ to every $F \in C^(G)$, then*

$$\lim_{\delta \rightarrow 0} \text{dis}(\pi_{\varepsilon(S^*-Q^*)}(F) - \sigma_{\varepsilon,-\varepsilon,\delta}(F), \mathcal{K}(L^2(\mathbb{R}^3))) = 0.$$

Here $\text{dis}(a, \mathcal{K}(\mathcal{H}))$ means the distance of an operator $a \in B(\mathcal{H})$ to the algebra $\mathcal{K}(\mathcal{H})$ of compact operators on the Hilbert space \mathcal{H} .

Proof. We have that

$$\left\| \sum_{i=1}^2 \sigma_{\varepsilon S^*}(F) \circ M_{\tilde{\varphi}_i} \circ M_{\delta,i} \right\|_{\text{op}} \leq 2\|\sigma_{\varepsilon S^*}(F)\|_{\text{op}}.$$

Similarly, we have $\left\| \sum_{j=\pm 1} \sigma_{j\sqrt{2}R^*-2Q^*}(F) \circ M_{\delta,4,\pm} \right\|_{\text{op}} \leq 2\|\sigma_{\varepsilon R^*}(F)\|_{\text{op}}$, since $\pm\sqrt{2}R^* - 2Q^*$ is contained in the G -orbit of $\pm R^*$. Together with Proposition 6.6, the inequality holds.

The kernel function $D_{\varepsilon,-\varepsilon,\delta,1}$ of the operator $\pi_{\varepsilon(S^*-Q^*)}(F) - \sigma_{\varepsilon S^*}(F) \circ M_{\tilde{\varphi}_1} \circ M_{\delta,1}$ is given by

$$\begin{aligned} & D_{\varepsilon,-\varepsilon,\delta,1}((s,t,v)(a,b,p)) \\ &= e^{\frac{1}{2}(a-b)} \left(\widehat{F}^{\mathfrak{h}}(s-a, t-b, e^{\frac{a-b}{2}}(v-p), \varepsilon(-e^{-b} + \frac{e^{-b}p^2}{2})Q^* - (e^{-\frac{a+b}{2}}p)R^* + e^{-a}S^*) \right. \\ & \quad \left. - \widehat{F}^{\mathfrak{h}}(s-a, t-b, e^{\frac{a-b}{2}}(v-p), \varepsilon(e^{-b}\frac{p^2}{2})Q^* - (e^{-\frac{a+b}{2}}p)R^* + e^{-a}S^*)\varphi(-e^{-b})1_{S_{\delta,1}}(a,b,p) \right). \end{aligned}$$

Hence

$$\begin{aligned}
& \|(\pi_{\varepsilon, -\varepsilon}(F) - \sigma_{\varepsilon S^*}(F) \circ M_{\tilde{\varphi}_1} \circ M_{\delta, 1}) \circ M_{\delta, 1}\|_{\text{H-S}}^2 \\
&= \int_{\mathbb{R}^5 \times \{a; e^{-a} > \delta^6\}} |(\widehat{F}^h(s, t, v, \varepsilon(-e^{-b} + \frac{e^{-b}p^2}{2})Q^* - (e^{-\frac{a+b}{2}}p)R^* + e^{-a}S^*) \\
&\quad - \widehat{F}^h(s, t, v, \varepsilon(e^{-b}\frac{p^2}{2})Q^* - (e^{-\frac{a+b}{2}}p)R^* + e^{-a}S^*)\varphi(-e^{-b})|^2 e^{a-b} ds dt dv da db dp \\
&= \int_{\mathbb{R}^5 \times \{a; e^{-a} > \delta^6\}} |(\widehat{F}^h(s, t, v, \varepsilon(-e^{-b} + \frac{e^a p^2}{2})Q^* - pR^* + e^{-a}S^*) \\
&\quad - \widehat{F}^h(s, t, v, \varepsilon(e^a \frac{p^2}{2})Q^* - pR^* + e^{-a}S^*)\varphi(-e^{-b})|^2 e^{\frac{3a-b}{2}} ds dt dv da db dp \\
&\leq \int_{\mathbb{R}^5 \times \{a; e^{-a} > \delta^6\}} 2|(\widehat{F}^h(s, t, v, \varepsilon(-e^{-b} + \frac{e^a p^2}{2})Q^* - pR^* + e^{-a}S^*)) (1 - \varphi(-e^{-b}))|^2 e^{\frac{3a-b}{2}} ds dt dv da db dp \\
&\quad + \int_{\mathbb{R}^5 \times \{a; e^{-a} > \delta^6\}} 2|(\widehat{F}^h(s, t, v, \varepsilon(-e^{-b} + \frac{e^a p^2}{2})Q^* - pR^* + e^{-a}S^*) \\
&\quad - \widehat{F}^h(s, t, v, \varepsilon(e^a \frac{p^2}{2})Q^* - pR^* + e^{-a}S^*))\varphi(-e^{-b})|^2 e^{\frac{3a-b}{2}} ds dt dv da db dp \\
&\leq \int_{\mathbb{R}^4 \times \{b; e^{-b} > \frac{1}{2}\} \times \{a; e^{-a} > \delta^6\}} |\alpha(s, t, v)\beta(-e^{-b} + \frac{e^a p^2}{2}, p, e^{-a})|^2 e^{\frac{3a-b}{2}} ds dt dv da db dp \\
&\quad + \int_{\mathbb{R}^4 \times \{b; b > -C\} \times \{a; e^{-a} > \delta^6\}} e^{-2b} |\alpha(s, t, v)\beta(\frac{e^a p^2}{2}, p, e^{-a})|^2 e^{\frac{3a-b}{2}} ds dt dv da db dp \\
&< \infty
\end{aligned}$$

for two continuous functions α, β on \mathbb{R}^3 with compact support and some $C > 0$. This shows that the operator $(\pi_{\varepsilon}(S^* - Q^*) - \sigma_{\varepsilon S^*}(F) \circ M_{\tilde{\varphi}_1} \circ M_{\delta, 1}) \circ M_{\delta, 1}$ is Hilbert-Schmidt, therefore, it is compact.

Let us estimate the operator norm of the linear endomorphism

$$D_{\varepsilon, \delta, 3, l, +} := (\pi_{\varepsilon, -\varepsilon}(F) - N_{\delta, 3, l} \circ \sigma_{\varepsilon(-1 + \frac{\delta^4 l^2}{2})Q^*}) \circ M_{\delta, 3, l, +}.$$

We have seen in relation (6.1.2) that

$$\pi_{\varepsilon, -\varepsilon}(F) \circ M_{\delta, 3, l, +} = N_{\delta, 3, l} \circ \pi_{\varepsilon, -\varepsilon}(F) \circ M_{\delta, 3, l, +}.$$

The kernel function $F_{\varepsilon, \delta, 3, l, +}$ of $D_{\varepsilon, \delta, 3, l, +}$ is given by

$$\begin{aligned}
& F_{\varepsilon, \delta, 3, l, +}(s, t, v, a, b, p) \\
&= \left(\widehat{F}^h(s - a, t - b, e^{\frac{a-b}{2}}(v - p), \varepsilon(e^{-b}(-1 + \frac{p^2}{2})Q^* - (e^{-\frac{a+b}{2}}p)R^* + e^{-a}S^*) \right. \\
&\quad \left. - \widehat{F}^h(s - a, t - b, e^{\frac{a-b}{2}}(v - p), \varepsilon(e^{-b}(-1 + \frac{\delta^4 l^2}{2})Q^*)) \right) \\
&\quad e^{\frac{1}{2}(a-b)} 1_{S_{\delta, 3, l, +}}(a, b, p) 1_{T_{\delta, 3, l}}(s, t, v).
\end{aligned}$$

On the set $S_{\delta,3,l,+}$, we have that $\frac{1}{\delta^{3/4}} \geq e^{-b} \geq \delta$, $e^{-b/2}|1 - \frac{p^2}{2}| > \delta^{1/2}$ and $|p - l\delta^2| < \delta^2$. Since $F \in L_c^1$, there exists $\rho > 0$ such that $\widehat{F}^b(s, t, v, qQ^* + rR^* + zS^*) = 0$ whenever $|q| > \rho$ or $|r| > \frac{1}{2}\rho$.

If now $e^{-b/2} > \frac{1}{\delta^{3/4}} \geq \frac{\rho}{\delta^{1/2}}$ (for δ small enough), then

$$\widehat{F}^b(s - a, t - b, e^{\frac{a-b}{2}}(v - p), \varepsilon(e^{-b}(-1 + \frac{p^2}{2})Q^* - (e^{-\frac{a+b}{2}}p)R^* + e^{-a}S^*)) = 0.$$

If p is large, then $e^{-b}(\frac{p^2}{2} - 1) > \rho$ implying $F_{\varepsilon,\delta,3,l,+}(s, t, v, a, b, p) = 0$ tells us that we can assume that $e^{-b}|p| \leq e^{-b}(\frac{p^2}{2} - 1) \leq \rho$ for every p giving a nonzero value to the function $F_{\delta,3,k,+}$. For $|p|$ small, we have then $e^{-b}(|p| + |l|\delta^2) \leq \frac{C}{\delta^{1/2}}$.

Since $e^{-a} \leq \delta^6$ in the set $S_{\delta,3,l,+}$, it follows that $e^{-\frac{a+b}{2}}|p| < \delta$ whenever $F_{\delta,3,l,+}(-, e^{-\frac{a+b}{2}}p, -) \neq 0$. According to Remark 6.1 we can now write

$$\begin{aligned} & |\widehat{F}^b(s - a, t - b, e^{\frac{a-b}{2}}(v - p), \varepsilon(-e^{-b} + \frac{e^{-b}p^2}{2})Q^* - (e^{-\frac{a+b}{2}}p)R^* + e^{-a}S^*) \\ & - \widehat{F}^b(s - a, t - b, e^{\frac{a-b}{2}}(v - p), \varepsilon(-e^{-b} + e^{-b}\frac{\delta^4 l^2}{2})Q^*)| \\ & \leq (e^{-b}|\frac{\delta^4 l^2 - p^2}{2}| + |e^{-\frac{a-b}{2}}p| + e^{-a})\psi(s - a, t - b, e^{\frac{a-b}{2}}(v - p)) \end{aligned}$$

for some non-negative C^∞ -function ψ with compact support contained in $[-M, M]^3$. Hence for any $(a, b, p) \in \mathbb{R}^3$:

$$|F_{\varepsilon,\delta,3,l,+}(s, t, v, a, b, p)| \leq M\delta\psi(s - a, t - b, e^{\frac{a-b}{2}}(v - p))e^{\frac{1}{2}(a-b)},$$

for all $a, b, p, s, t, v \in \mathbb{R}$ and some $M > 0$ independent of l, F , and δ small enough. According to the Young's inequality, this tells us that

$$\|(\pi_{\varepsilon,-\varepsilon}(F) - N_{\delta,3,l} \circ \sigma_{\varepsilon(-1+\frac{\delta^4 l^2}{2})}Q^*) \circ M_{\delta,3,l,+}\|_{\text{op}} \leq M\delta\|\psi\|_1,$$

for $l \in \mathbb{Z}$, $F \in L_c^1$ and $\delta > 0$ small enough.

We have seen in (6.1.1) that $F_{\varepsilon,\delta,3,l,+}(s, t, v, a, b, p) = 0$ whenever $(s, t, v) \notin T_{\delta,3,l}$. Consequently for any sufficiently small $1 > \delta > 0$ we see that

$$\|(\pi_{\varepsilon,-\varepsilon}(F) - \sigma_{\delta,3}(F)) \circ (M_{\delta,3,+} + M_{\delta,3,-})\|_{\text{op}} \leq C\delta$$

for some constant $C > 0$.

For $i = 2$, the kernel function $D_{\delta,2}$ of the operator $(\pi_{\varepsilon,-\varepsilon}(F) - \sigma_{\varepsilon}S^*(F)) \circ M_{\delta,2}$ is given by

$$\begin{aligned} & D_{\delta,2}(s, t, v, a, b, p) \\ & = e^{\frac{1}{2}(a-b)} \left(\widehat{F}^b(s - a, t - b, e^{\frac{a-b}{2}}(v - p), \varepsilon(-e^{-b} + \frac{e^{-b}p^2}{2})Q^* + (-e^{-\frac{a+b}{2}}p)R^* + e^{-a}S^*) \right. \\ & \quad \left. - \widehat{F}^b(s - a, t - b, e^{\frac{a-b}{2}}(v - p), \varepsilon e^{-b}\frac{p^2}{2}Q^* - e^{-\frac{a+b}{2}}pR^* + e^{-a}S^*) \right) 1_{S_{\delta,2}}(a, b, p). \end{aligned}$$

Then we have a similar manipulation:

$$\begin{aligned}
& |D_{\delta,2}(s, t, v, a, b, p)| \\
& \leq e^{\frac{1}{2}(a-b)} \left| \left(\widehat{F}^{\mathfrak{h}}(s-a, t-b, e^{\frac{a-b}{2}}(v-p), \varepsilon(-e^{-b} + \frac{e^{-b}p^2}{2})Q^* + (-e^{-\frac{a+b}{2}}p)R^* + e^{-a}S^*) \right. \right. \\
& \quad \left. \left. - \widehat{F}^{\mathfrak{h}}(s-a, t-b, e^{\frac{a-b}{2}}(v-p), (\varepsilon e^{-b}\frac{p^2}{2})Q^* + (-e^{-\frac{a+b}{2}}p)R^* + e^{-a}S^*) \right) \right| 1_{S_{\delta,2}}(a, b, p) \\
& \leq e^{-b} e^{\frac{1}{2}(a-b)} |\psi(s-a, t-b, e^{\frac{a-b}{2}}(v-p))| 1_{S_{\delta,2}}(a, b, p) \\
& \leq \delta e^{\frac{1}{2}(a-b)} \psi(s-a, t-b, e^{\frac{a-b}{2}}(v-p))
\end{aligned}$$

for some $\psi \in C_c(\mathbb{R}^3)$. Again we have the estimate

$$\|(\pi_{\varepsilon, -\varepsilon}(F) - \sigma_{\varepsilon S^*}(F)) \circ M_{\delta,2}\|_{\text{op}} \leq \delta \int_{\mathbb{R}^3} |\psi(a, b, p)| da db dp. \quad (6.1.4)$$

Finally for $i = 4$, the kernel function $D_{\delta,4,k,\pm}$ of the operator $(\pi_{\varepsilon, -\varepsilon}(F) - \sigma_{\varepsilon(\pm\sqrt{2}R^* - 2Q^*)}(F)) \circ M_{\delta,4,\pm}$ is given by

$$\begin{aligned}
& D_{\delta,4,\pm}(s, t, v, a, b, p) \\
& = e^{\frac{1}{2}(a-b)} \left(\widehat{F}^{\mathfrak{h}}(s-a, t-b, e^{\frac{a-b}{2}}(v-p), \varepsilon(-e^{-b} + \frac{e^{-b}p^2}{2})Q^* - \varepsilon(e^{-\frac{a+b}{2}}p)R^* + \varepsilon e^{-a}S^*) \right. \\
& \quad \left. - \widehat{F}^{\mathfrak{h}}(s-a, t-b, e^{\frac{a-b}{2}}(v-p), -\varepsilon(e^{-b}(\pm\sqrt{2}p+2))Q^* + \varepsilon(e^{-\frac{a+b}{2}}(\pm\sqrt{2}))R^*) \right) 1_{S_{\delta,4,\pm}}(a, b, p).
\end{aligned}$$

Then we have as in the preceding case on the set $S_{\varepsilon,\delta,4,\pm}$ that

$$e^{-b/2} \left| \frac{p}{\sqrt{2}} \pm 1 \right| \leq e^{-b/2} \left| \frac{p^2}{2} - 1 \right| \leq \delta^{1/2},$$

and

$$e^{-b} \left(\frac{p \pm \sqrt{2}}{2} \right)^2 \leq \frac{1}{2} e^{-b} \left(\frac{p^2}{2} - 1 \right)^2.$$

$$\begin{aligned}
& |D_{\delta,4,k,+}(s, t, v, a, b, p)| \\
& \leq e^{\frac{1}{2}(a-b)} \left| \widehat{F}^{\mathfrak{h}}(s-a, t-b, e^{\frac{a-b}{2}}(v-p), \varepsilon(-e^{-b} + \frac{e^{-b}p^2}{2})Q^* + \varepsilon(-e^{-\frac{a+b}{2}}p)R^* + e^{-a}S^*) \right. \\
& \quad \left. - \widehat{F}^{\mathfrak{h}}(s-a, t-b, e^{\frac{a-b}{2}}(v-p), -\varepsilon e^{-b}(2\pm\sqrt{2}p)Q^* + \varepsilon(e^{-\frac{a+b}{2}}(\pm\sqrt{2}))R^*) \right| 1_{S_{\delta,4,\pm}}(a, b, p) \\
& \leq (|e^{-b}(-1 + \frac{p^2}{2} \pm \sqrt{2}p + 2)| + e^{-\frac{a+b}{2}}|p \pm \sqrt{2}| + e^{-a}) \\
& \quad e^{\frac{1}{2}(a-b)} \psi(s-a, t-b, e^{\frac{a-b}{2}}(v-p)) 1_{S_{\delta,4,\pm}}(a, b, p) \\
& \leq (e^{-b} \frac{(p \pm \sqrt{2})^2}{2} + e^{-\frac{a+b}{2}}|p \pm \sqrt{2}| + e^{-a}) e^{\frac{1}{2}(a-b)} \psi(s-a, t-b, e^{\frac{a-b}{2}}(v-p)) 1_{S_{\delta,4}}(a, b, p) \\
& \leq (e^{-b}(\frac{p^2}{2} - 1)^2 + e^{-\frac{a+b}{2}}\sqrt{2}|\frac{p^2}{2} - 1| + e^{-a}) e^{\frac{1}{2}(a-b)} \psi(s-a, t-b, e^{\frac{a-b}{2}}(v-p)) 1_{S_{\delta,4}}(a, b, p) \\
& \leq (\delta + \sqrt{2}\delta^{7/2} + \delta^6) e^{\frac{1}{2}(a-b)} \psi(s-a, t-b, e^{\frac{a-b}{2}}(v-p)) 1_{S_{\delta,4,\pm}}(a, b, p).
\end{aligned}$$

Hence as before:

$$\|(\pi_{\varepsilon,-\varepsilon}(F) - \sigma_{-\varepsilon(\pm\sqrt{2}R^*+2Q^*)}(F)) \circ M_{\delta,4,\pm}\|_{\text{op}} \leq \delta \int_{\mathbb{R}^3} \psi(a,b,p) da db dp. \quad (6.1.5)$$

We have seen above that $(\pi_{\varepsilon,-\varepsilon}(F) - \sigma_{\varepsilon S^*}(F) \circ M_{\tilde{\varphi}_1}) \circ M_{\delta,1}$ is compact for any $\delta > 0$. Hence it follows from the estimates above that

$$\pi_{\varepsilon,-\varepsilon}(F)(1-M_{\delta,1}) - \left(\sigma_{\varepsilon S^*}(F) \circ M_{\delta,2} + \sum_{j=\pm} \sigma_{\varepsilon(j\sqrt{2}R^*-2Q^*)}(F) \circ M_{\delta,4,j} + \sum_{j=\pm} \sigma_{\delta,3,j}(F) \right)$$

converges to 0 as δ tends to 0. Therefore,

$$\lim_{\delta \rightarrow 0} \text{dis}(\pi_{\varepsilon(S^*-Q^*)}(F) - \sigma_{\varepsilon,-\varepsilon,\delta}(F), \mathcal{K}(L^2(\mathbb{R}^3))) = 0.$$

Since L_c^1 is dense in $C^*(G)$, the relation $\text{dis}(\pi_{\varepsilon(S^*-Q^*)}(F) - \sigma_{\varepsilon,-\varepsilon,\delta}(F), \mathcal{K}(L^2(\mathbb{R}^3))) \rightarrow 0$ as $\delta \rightarrow 0$ remains true for any $F \in C^*(G)$. \square

6.1.2 The orbits $\Omega_{\varepsilon,\varepsilon} = \Omega_{\varepsilon Q^* + \varepsilon S^*}, \varepsilon = \pm 1$

The boundary orbits of $\Omega_{\varepsilon,\varepsilon|_{\mathfrak{h}}} \subset \mathfrak{h}^*$ are the orbits of εS^* and of εQ^* inside \mathfrak{h}^* . Therefore we must decompose $\mathbb{R}^3 \simeq G/H$ into three disjoint subsets. Since this case is similar but much easier to the preceding case, we shall omit the proofs.

Definition 6.8. 1. For $k \in \mathbb{Z}$, denote by

$$J_{\delta,k} := \left[\log\left(\frac{1}{\delta^6}\right), +\infty \right) \times \left(-\infty, \log\left(\frac{1}{\delta}\right) \right] \times I_{\delta^2,k}$$

where $I_{c,l} := [cl, cl+c) \subset \mathbb{R}$ for $l \in \mathbb{Z}, c > 0$ and let

$$L_{\delta,k} := \mathbb{R} \times \mathbb{R} \times I_{\delta^2,k}.$$

2. Let

$$K_{\delta} = \left[-\frac{1}{\delta}, \frac{1}{\delta}\right] \times \left[-\frac{1}{\delta}, \frac{1}{\delta}\right] \times \left[-\frac{1}{\delta^{1/2}}, \frac{1}{\delta^{1/2}}\right].$$

3. Let $S_{\delta,1} := \{(a,b,p); e^{-a} > \delta^6\}$ (corresponds to $\mathbb{R}_{\varepsilon S^*}$).

4. Let $S_{\delta,2} := \{(a,b,p); e^{-a} \leq \delta^6, e^{-b} < \delta\}$ (corresponds to $\mathbb{R}_{\varepsilon S^*}$).

5. Let

$$S_{\delta,3,k} := J_{\delta,k}, \quad k \in \mathbb{Z}$$

(corresponds to $\mathbb{R}_{\varepsilon Q^*}$) and $S_{\delta,3} := \bigcup_k S_{\delta,3,k}$.

6. Define the corresponding multiplication operators on $L^2(\mathbb{R}^3)$ by $M_{\delta,i}$ for $i = 1, 2$ respectively, and for $k \in \mathbb{Z}$: $M_{\delta,3,k} = M_{S_{\delta,3,k}}, M_{\delta,3} = M_{S_{\delta,3}}$.

Remark 6.9.

1. For any $\delta > 0$, the sets $\{S_{\delta,1}, S_{\delta,2}, S_{\delta,3,k}; k \in \mathbb{Z}\}$ form a partition of \mathbb{R}^3 .
2. Recall that for $\alpha = (a, b, p)$ and $\beta = (s, t, v) \in \mathbb{R}^3$, the multiplication of $E(s, t, v) \cdot E(a, b, p)$ is given by

$$E(s, t, v) \cdot E(a, b, p) = E(s + a, t + b, e^{\frac{-a+b}{2}}v + p).$$

For $\beta \in K_\delta, \alpha \in S_{\delta,3,k}$, we have that $e^{-a} \leq \delta^6, e^b < \frac{1}{\delta}, |v| < \frac{1}{\delta^{1/2}}$ and so $e^{\frac{-a+b}{2}}|v| < \delta^2$ and

$$\begin{aligned} \beta \cdot \alpha &= (s + a, t + b, e^{\frac{-a+b}{2}}v + p) \\ &\in \mathbb{R} \times \mathbb{R} \times ([-\delta^2, \delta^2] + I_{\delta,k}) \\ &\subset \bigcup_{i=-1}^1 L_{\delta,k+i} =: T_{\delta,3,k}. \end{aligned} \tag{6.1.6}$$

Let $N_{\delta,3,k}$ be the multiplication operator with the characteristic function of the set $T_{\delta,3,k}$ for $k \in \mathbb{Z}$. We see that the set $T_{\delta,3,k}$ is contained in the union of 3 disjoint boxes $L_{\delta,k+i}$ for $i \in \{-1, 0, 1\}$. Hence it follows from the relations (6.1.6) that for $\delta > 0$ small enough,

$$\widehat{F}^h(st^{-1}, t \cdot (\varepsilon(S^* + Q^*))) = 0 \text{ for } t \in S_{\delta,3,k}, s \notin T_{\delta,3,k} \text{ and } k \in \mathbb{Z}.$$

Hence,

$$N_{\delta,3,k} \circ \pi_{\varepsilon,\varepsilon}(F) \circ M_{\delta,3,k} = \pi_{\varepsilon,\varepsilon}(F) \circ M_{\delta,3,k} \text{ for } k \in \mathbb{Z}. \tag{6.1.7}$$

The representation $\pi_{\varepsilon,\varepsilon} = \text{ind}_H^G \chi_\varepsilon(Q^* + S^*)$ acts on $\xi \in L^2(G/H, \chi_\varepsilon(Q^* + S^*))$ in the following way:

$$\begin{aligned} &\pi_{\varepsilon,\varepsilon}(F)\xi(u) \\ &= \int_{\mathbb{R}^3} e^{\frac{1}{2}(a-b)} \widehat{F}^h(s - a, t - b, e^{\frac{a-b}{2}}(v - p), \varepsilon(e^{-b}(1 + \frac{p^2}{2})Q^* + (-e^{-\frac{a+b}{2}}p)R^* + e^{-a}S^*)) \\ &\quad \xi(E(a, b, p)) da db dp. \end{aligned}$$

We define the linear operator $\sigma_{\varepsilon,\varepsilon,\delta}(F)$ by the rule

$$\sigma_{\varepsilon,\varepsilon,\delta}(F) := \sigma_{\varepsilon S^*}(F) \circ M_{\widehat{\varphi}_1} \circ M_{\delta,1} + \sigma_{\varepsilon S^*}(F) \circ M_{\delta,2} + \sigma_{\delta,3}(F), \tag{6.1.8}$$

where $\sigma_{\delta,3}(F) := \sum_{k \in \mathbb{N}} N_{\delta,k,3} \circ \sigma_{\varepsilon(1 + \frac{\delta^4 k^2}{2})Q^*}(F) \circ M_{\delta,3,k}$, the operator $M_{\widehat{\varphi}_1}$ is the multiplication operator with the function $\widehat{\varphi}_1(a, b, p) := \varphi(e^{-b})$ where $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is a C^∞ -function with support contained in $[-1, 1]$ such that $\varphi(0) = 1$ and $\|\varphi\|_\infty = 1$. The kernel functions $F_{\delta,i}$, for $i = 1, 2, 3$, of these operators are given

by

$$\begin{aligned}
F_{\delta,1}((s,t,v),(a,b,p)) &:= \widehat{F}^{\mathfrak{h}}(s-a, t-b, e^{\frac{a-b}{2}}(v-p), \varepsilon((\frac{e^{-b}p^2}{2})Q^* + (-e^{-\frac{a+b}{2}}p)R^* + e^{-a}S^*)) \\
&\quad e^{\frac{1}{2}(a-b)}\varphi(e^{-b})1_{S_{\delta,1}}(a,b,p), \\
F_{\delta,3}((s,t,v),(a,b,p)) &= \sum_{k \in \mathbb{N}} \left(\widehat{F}^{\mathfrak{h}}(s-a, t-b, e^{\frac{a-b}{2}}(v-p), \varepsilon e^{-b}(1 + \frac{\delta^4 k^2}{2})Q^*) \right) \\
&\quad e^{\frac{1}{2}(a-b)}1_{S_{\delta,k,3}}(a,b,p), \\
F_{\delta,2}((s,t,v),(a,b,p)) &:= \widehat{F}^{\mathfrak{h}}(s-a, t-b, e^{\frac{a-b}{2}}(v-p), \varepsilon(\frac{e^{-b}p^2}{2})Q^* + (-e^{-\frac{a+b}{2}}p)R^* + e^{-a}S^*)) \\
&\quad e^{\frac{1}{2}(a-b)}1_{S_{\delta,2}}(a,b,p).
\end{aligned}$$

Now the representation $\sigma_{\varepsilon(1+\frac{k^2\delta^4}{2})Q^*}$ is equivalent to the representation $\sigma_{\varepsilon Q^*}$, since both linear functionals are in the same G -orbit. Therefore, for any $k \geq 0$ and $\delta > 0$,

$$\|\sigma_{\varepsilon(1+\frac{\delta^4 k^2}{2})Q^*}(F) \circ M_{\delta,3,k}\|_{\text{op}} \leq \|\sigma_{\varepsilon Q^*}(F)\|_{\text{op}}. \quad (6.1.9)$$

It follows from Proposition 6.6 that there exists $C > 0$ such that the operator $\sigma_{\delta,3}(F)$ is bounded by C for any $F \in L_c^1$.

The proof of the following proposition is similar but much easier than that of Proposition 6.7 and will be left to the reader.

Proposition 6.10. *For any $F \in L_c^1$ and $\delta > 0$, we have that*

$$\|\sigma_{\varepsilon,\varepsilon,\delta}(F)\|_{\text{op}} \leq 2\|\sigma_{\varepsilon S^*}(F)\|_{\text{op}} + C\|\sigma_{\varepsilon Q^*}(F)\|_{\text{op}}$$

and we can extend the mapping $F \mapsto \sigma_{\varepsilon,\varepsilon,\delta}(F)$ to every $F \in C^*(G)$, then

$$\lim_{\delta \rightarrow 0} \text{dis}(\pi_{\varepsilon(Q^*+S^*)}(F) - \sigma_{\varepsilon,\varepsilon,\delta}(F), \mathcal{K}(L^2(\mathbb{R}^3))) = 0.$$

6.2 The boundary condition for $\mathbb{R}_{\varepsilon S^*}$, $\varepsilon = \pm 1$

In this subsection, we consider the subset $\Gamma_5 = \{\mathbb{R}_{\varepsilon S^*}, \varepsilon = \pm 1\}$ of \widehat{G} . We use the coordinates of G according to the basis $\{X = A+B, B, P, Q, R, S\}$ of \mathfrak{g} . We recall that the boundary of the orbit of $b^*B^* + \varepsilon S^*$ is the orbit of the functionals $aA^* + \varepsilon Q^*$, $x^*X^* \pm P^*$ and $a^*A^* + b^*B^*$ for $a^*, b^*, x^* \in \mathbb{R}$. This tells us that we must find the conditions from these boundary points.

Definition 6.11. *Let*

$$\sigma_{\varepsilon S^*} := \text{ind}_H^G \chi_{\varepsilon S^*}.$$

The kernel function \mathcal{K}_F of the operator $\sigma_{\varepsilon S^*}(F)$ is given by

$$\mathcal{K}_F(s, t, v; x, b, p) = \widehat{F}^{\mathfrak{h}}(s-x, t-b, e^{\frac{-b}{2}}(v-p), \varepsilon(\frac{e^{-x-b}p^2}{2})Q^* - \varepsilon(e^{-\frac{2x+b}{2}}p)R^* + \varepsilon e^{-x}S^*)e^{-b/2}.$$

This representation is equivalent to the direct integral representation

$$\tau_{\varepsilon S^*} := \int_{\mathbb{R}}^{\oplus} \pi_{b^* B^* + \varepsilon S^*} db^*$$

acting on the Hilbert space

$$\mathcal{H}_{\tau_{\varepsilon S^*}} = \int_{\mathbb{R}}^{\oplus} L^2(G/\exp(\mathbb{R}B + \mathfrak{h}), \chi_{b^* B^* + \varepsilon S^*}) db^* \simeq \int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}^2) db^*$$

with the norm

$$\|\xi\|_2^2 = \int_{\mathbb{R}} \|\xi(b^*)\|_2^2 db^* \quad \text{for } \xi \in \mathcal{H}_{\tau_{\varepsilon S^*}}.$$

An intertwining operator $U_{\varepsilon S^*}$ for this equivalence is given by

$$U_{\varepsilon S^*}(\xi)(b^*)(g) := \int_{\mathbb{R}} \xi(g \exp(bB)) e^{\frac{1}{4}b} e^{-2\pi i b^* b} db, \quad \xi \in L^2(G/H, \chi_{\varepsilon S^*}), g \in G, b^* \in \mathbb{R}.$$

Let $C(\mathbb{R}, \mathcal{B})$ be the C^* -algebra of all continuous uniformly bounded mappings from \mathbb{R} into the algebra \mathcal{B} of bounded operators on the Hilbert space $L^2(\mathbb{R})$, containing the ideal $C_0(\mathbb{R}, \mathcal{K})$ of all continuous mappings from \mathbb{R} into the algebra of compact operators on $L^2(\mathbb{R})$ vanishing at infinity. It follows from Section 5 that the image of $\tau_{\varepsilon S^*}$ is contained in $C(\mathbb{R}, \mathcal{B})$. On the other hand, we can consider $C(\mathbb{R}, \mathcal{B})$ as a subalgebra of $B(\mathcal{H}_{\tau_{\varepsilon S^*}})$ as for $\phi \in C(\mathbb{R}, \mathcal{B})$, let

$$\phi(\xi)(b^*) := \phi(b^*)(\xi(b^*)) \quad \text{for } \xi \in \mathcal{H}_{\tau_{\varepsilon S^*}}, b^* \in \mathbb{R}.$$

The unitary mapping $U_{\varepsilon S^*}$ induces a canonical homomorphism $\rho_{\varepsilon S^*}$ from the algebra $B(L^2(G/H, \chi_{\varepsilon S^*}))$ onto $B(\mathcal{H}_{\tau_{\varepsilon S^*}})$. This homomorphism is defined on $\sigma_{\varepsilon S^*}(a)$ by

$$\begin{aligned} \rho_{\varepsilon S^*}(\sigma_{\varepsilon S^*}(a)) &= U_{\varepsilon S^*} \circ \sigma_{\varepsilon S^*}(a) \circ U_{\varepsilon S^*}^* \\ &= \int_{\mathbb{R}}^{\oplus} \pi_{b^* B^* + \varepsilon S^*}(a) db^* \\ &= \tau_{\varepsilon S^*}(a) \quad \text{for } a \in C^*(G). \end{aligned} \tag{6.2.1}$$

Definition 6.12. Let $\partial \mathbb{R}_{\varepsilon S^*}$ be the boundary of $\mathbb{R}_{\varepsilon S^*}$ in \mathfrak{g}^*/G .

It is easy to see from the relations (3.1.1) that $\partial \mathbb{R}_{\varepsilon S^*} = \{Q, R, S\}^{\perp} + (\mathbb{R}^{+,0}_{\varepsilon})Q^* \subset S^{\perp}$.

Definition 6.13. 1. For $k \in \mathbb{Z}$, denote by

$$J_{\delta,k} := \left[\log\left(\frac{1}{\delta^6}\right), +\infty \right) \times \mathbb{R} \times I_{\delta^2,k},$$

where $I_{c,l} := [cl, cl+c) \subset \mathbb{R}$ for $l \in \mathbb{Z}$ and $c > 0$.

2. Let $S_{\delta,1} := \{(x, b, p); e^{-x} > \delta^6\}$.

3. Let $\delta \mapsto r_\delta \in \mathbb{R}^+$ be such that $\lim_{\delta \rightarrow 0} r_\delta = +\infty$ and $\lim_{\delta \rightarrow 0} e^{r_\delta} \delta^{1/2} = 0$.

4. For a constant $D > 0$ and $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$, let

$$S_{\delta,D,3,k} := \{(x, b, p) \in \mathbb{R}^3 : e^{-x} \leq \delta^6, x \in I_{r_\delta, k_1}, b \in I_{r_\delta, k_2}, p \in I_{D\delta^2 e^{r_\delta/2(k_1+k_2)}, k_3}\}.$$

Proposition 6.14. *For every compact subset $K \subseteq \mathbb{R}^3$, we have that*

$$KS_{\delta,D,3,k} \subset \bigcup_{\substack{j \in \mathbb{Z}^3 \\ |j_i| \leq 1, i=1,2,3}} S_{\delta, D e^{\frac{r_\delta}{2}(-j_1-j_2)}, 3, k+j} =: R_{\delta,D,3,k}$$

for every $k \in \mathbb{Z}^3$ and $\delta > 0$ small enough.

Proof. Indeed, we have an $M > 0$ such that $K \subset [-M, M]^3$ and then for $r_\delta > M$ and $(s, t, v) \in K, (x, b, p) \in S_{\delta,D,3,k}$, it follows that

$$u := (s, t, v) \cdot (x, b, p) = (s + x, t + b, e^{b/2}v + p),$$

$(k_1 + j_1)r_\delta \leq s + x < (k_1 + j_1 + 1)r_\delta$ and $(k_2 + j_2)r_\delta \leq t + b < (k_2 + j_2 + 1)r_\delta$ for some $j_1, j_2 \in \{-1, 0, 1\}$. It follows that

$$\begin{aligned} |e^{b/2}v| &\leq |e^{-x/2}v|e^{(x+b)/2} \\ &\leq \delta^3 |v| e^{(x+b)/2} \\ &\leq \delta^3 |v| e^{r_\delta/2(-j_1-j_2)} e^{r_\delta/2(k_1+k_2+j_1+j_2)} e^{r_\delta} \\ &< D e^{r_\delta/2(-j_1-j_2)} \delta^2 e^{r_\delta/2(k_1+k_2+j_1+j_2)} \end{aligned}$$

for δ small enough, since $\lim_{\delta \rightarrow 0} e^{r_\delta} \delta^{1/2} = 0$. Hence,

$$\begin{aligned} p + e^b v &< (k_3 + 1) D e^{\frac{r_\delta}{2}(-j_1-j_2)} \delta^2 e^{\frac{r_\delta}{2}(k_1+k_2+j_1+j_2)} + e^b v \\ &< (k_3 + 2) D e^{\frac{r_\delta}{2}(-j_1-j_2)} \delta^2 e^{\frac{r_\delta}{2}(k_1+k_2+j_1+j_2)} \end{aligned}$$

and also

$$\begin{aligned} p + e^b v &\geq k_3 D e^{\frac{r_\delta}{2}(-j_1-j_2)} \delta^2 e^{\frac{r_\delta}{2}(k_1+k_2+j_1+j_2)} - e^b |v| \\ &\geq (k_3 - 1) D e^{\frac{r_\delta}{2}(-j_1-j_2)} \delta^2 e^{\frac{r_\delta}{2}(k_1+k_2+j_1+j_2)}. \end{aligned}$$

Hence u is contained in $R_{\delta,D,3,k}$. \square

We define the multiplication operators $M_{\delta,1} = M_{S_{\delta,1}}$, respectively, $M_{\delta,D,3,k} = M_{S_{\delta,D,3,k}}$ and $P_{\delta,D,3,k} := M_{R_{\delta,D,3,k}}$ on $L^2(\mathbb{R}^3)$. Then we have the following proposition.

Proposition 6.15. *For $\delta > 0$,*

$$\overline{M}_{\delta,1} \circ U_{\varepsilon S^*} = U_{\varepsilon S^*} \circ M_{\delta,1},$$

where $\overline{M}_{\delta,1} = M_{\{(x,p); e^{-x} > \delta^6\}}$ on $L^2(\mathbb{R}^2)$.

Proof. Indeed, for $\xi \in L^2(G/H, \chi_{S^*})$, $b \in \mathbb{R}$ and $g \in G$, we have that

$$\begin{aligned}
U_{S^*}(M_{\delta,1}(\xi)(E(x,p))) &= \int_{\mathbb{R}} M_{\delta,1}(\xi)(E(x,p) \exp(bB)) e^{\frac{1}{4}b} e^{-2\pi i b^* b} db \\
&= \int_{\mathbb{R}} 1_{\{e^{-x} > \delta^6\}} \xi(E(x,p) \exp(bB)) e^{\frac{1}{4}b} e^{-2\pi i b^* b} db \\
&= 1_{\{e^{-x} > \delta^6\}}(E(x,p)) \int_{\mathbb{R}} \xi(E(x,p) \exp(bB)) e^{\frac{1}{4}b} e^{-2\pi i b^* b} db \\
&= \overline{M}_{\delta,1}(U_{S^*}\xi)(E(x,p)),
\end{aligned}$$

here $E(x,p) = \exp(xX) \exp(pP)$ for $x, p \in \mathbb{R}$. \square

Definition 6.16. For $D > 0$, we define the linear operator $\sigma_{\varepsilon,\delta,D}(F)$ on $L^2(G/H, \chi_{\varepsilon S^*}) = L^2(\mathbb{R}^3)$ by the rule

$$\sigma_{\varepsilon,\delta,D}(F) := \sum_{k \in \mathbb{Z}^3} P_{\delta,D,3,k} \circ \sigma_{(k_3^2 \delta^3 D^2 e^{r_\delta(k_1+k_2)})_{\varepsilon Q^*}}(F) \circ M_{\delta,D,3,k}.$$

Now the representation $\sigma_{(k_3^2 \delta^3 D^2 e^{r_\delta(k_1+k_2)})_{\varepsilon Q^*}}$ is equivalent to the representation $\sigma_{\varepsilon Q^*}$, since both linear functionals are in the same G -orbit. Therefore for any k, δ , we have

$$\|\sigma_{(k_3^2 \delta^3 D^2 e^{r_\delta(k_1+k_2)})_{\varepsilon Q^*}}(F) \circ M_{\delta,D,3,k}\|_{\text{op}} \leq \|\sigma_{\varepsilon Q^*}(F)\|_{\text{op}}. \quad (6.2.2)$$

Proposition 6.17. For every $F \in C^*(G)$ and $\delta > 0$, the operator $\sigma_{\varepsilon,\delta,D}(F)$ is bounded in the operator norm by $3^3 \|\sigma_{\varepsilon Q^*}(F)\|_{\text{op}}$.

Proof. For $k \in \mathbb{Z}^3$, decompose the set $R_{\delta,D,3,k}$ into a disjoint union of measurable subsets $R_{k,j}$, $j = (j_1, j_2, j_3) \in \mathbb{Z}^3$ with $|j_i| \leq 1$, $i = 1, 2, 3$, such that $R_{k,j} \subset S_{\delta, D e^{\frac{r_\delta}{2}(-j_1-j_2)}, 3, k+j}$ for every j . This gives us at most 3^3 such subsets $R_{k,j}$ for fixed k . These sets $R_{k,j}$ are disjoint in k for fixed j , since the sets $S_{\delta, D e^{\frac{r_\delta}{2}(-j_1-j_2)}, 3, k+j}$ are mutually disjoint in k for fixed j . It suffices then to apply Proposition 6.2. \square

Proposition 6.18. Let $a \in C^*(G)$. For any $\delta > 0$, the element $\tau_{\varepsilon S^*}(a) \circ \overline{M}_{\delta,1} := \int_{\mathbb{R}}^{\oplus} \pi_{b^* B^* + \varepsilon S^*}(a) \circ \overline{M}_{\delta,1} db^*$ is in $C_0(\mathbb{R}, \mathcal{K})$.

Proof. We must prove first that the operators $\pi_{b^* B^* + \varepsilon S^*}(a) \circ \overline{M}_{\delta,1}$ for all $b^* \in \mathbb{R}$ and $\delta > 0$ are all compact. The kernel function F_{b^*} of the operator $\pi_{b^* B^* + \varepsilon S^*}(F) \circ \overline{M}_{\delta,1}$ is given by Formula (4.1.3):

$$\begin{aligned}
F_{b^*}((t,u), (x,p)) &= 1_{\{e^{-x} > \delta^6\}} \int_{\mathbb{R}} \widehat{F^{\mathfrak{h}}}(E(t-x, u - e^{-\frac{b}{2}} p) \exp(bB)) \\
&\quad \left(\varepsilon e^{-x} \left(\frac{1}{2} p^2 Q^* - p R^* + S^* \right) \right) e^{i b^* b} e^{\frac{b}{4}} db.
\end{aligned}$$

Hence for $F \in L_c^1$, using Remark 6.1:

$$\begin{aligned}
\|\pi_{b^*B^*+\varepsilon S^*}(F) \circ \overline{M}_{\delta,1}\|_{H-S}^2 &= \int_{\mathbb{R}^4} 1_{\{e^{-x} > \delta^6\}} |F_{b^*}(t, u, x, p)|^2 dx dt dudp \\
&= \int_{\mathbb{R}^4} 1_{\{e^{-x} > \delta^6\}} \left| \int_{\mathbb{R}} \widehat{F^{\mathfrak{h}}}(E(t-x, u - e^{-\frac{b}{2}}p) \exp bB) \right. \\
&\quad \left. \left(\varepsilon e^{-x} \left(\frac{1}{2} p^2 Q^* - pR^* + S^* \right) \right) e^{ib^*b} e^{\frac{b}{4}} db \right|^2 dx dt dudp \\
&\leq \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^4} 1_{\{e^{-x} > \delta^6\}} |\widehat{F^{\mathfrak{h}}}(E(t-x, u - e^{-\frac{b}{2}}p) \exp bB) \right. \right. \\
&\quad \left. \left. \left(\varepsilon e^{-x} \left(\frac{1}{2} p^2 Q^* - pR^* + S^* \right) \right) \right|^2 dx dt dudp \right)^{\frac{1}{2}} e^{\frac{b}{4}} db \right)^2 \\
&= \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^4} 1_{\{e^{-x} > \delta^6\}} |\widehat{F^{\mathfrak{h}}}(E(t, u) \exp bB) \right. \right. \\
&\quad \left. \left. \left(\varepsilon \left(\frac{1}{2} e^x p^2 Q^* - pR^* + e^{-x} S^* \right) \right) \right|^2 e^x dx dt dudp \right)^{\frac{1}{2}} e^{\frac{b}{4}} db \right)^2 \\
&< \infty,
\end{aligned}$$

since the function

$$(t, u, x, p, b) \mapsto 1_{\{e^{-x} > \delta^6\}} \widehat{F^{\mathfrak{h}}}(E(t, u) \exp bB) \left(\varepsilon \left(\frac{1}{2} e^x p^2 Q^* - pR^* + e^{-x} S^* \right) \right) e^x e^{\frac{b}{4}}$$

has compact support.

Now we must show that $\lim_{b^* \rightarrow \infty} \|\pi_{b^*B^*+\varepsilon S^*}(F) \circ \overline{M}_{\delta,1}\|_{\text{op}} = 0$. The kernel function F_{b^*} of the operator $\pi_{b^*B^*+\varepsilon S^*}(F) \circ \overline{M}_{\delta,1}$ can be written, using partial integration, as

$$\begin{aligned}
F_{b^*}((t, u), (x, p)) &= 1_{\{e^{-x} > \delta^6\}} \int_{\mathbb{R}} \widehat{F^{\mathfrak{h}}}(E(t-x, u - e^{-\frac{b}{2}}p) \exp bB) \\
&\quad \left(\varepsilon e^{-x} \left(\frac{1}{2} p^2 Q^* - pR^* + S^* \right) \right) e^{ib^*b + \frac{b}{4}} db \\
&= 1_{\{e^{-x} > \delta^6\}} \int_{\mathbb{R}} \frac{1}{ib^* + \frac{1}{4}} (e^{-b/2} p \partial_2 + \partial_B) \widehat{F^{\mathfrak{h}}}(E(t-x, u - e^{-\frac{b}{2}}p) \exp bB) \\
&\quad \left(\varepsilon e^{-x} \left(\frac{1}{2} p^2 Q^* - pR^* + S^* \right) \right) e^{ib^*b} e^{\frac{b}{4}} db \\
&= 1_{\{e^{-x} > \delta^6\}} \int_{\mathbb{R}} \frac{1}{ib^* + \frac{1}{4}} (e^{-b/2} p \partial_2 + \partial_B) \widehat{F^{\mathfrak{h}}}(E(t-x, u - e^{-\frac{b}{2}}p) \exp bB) \\
&\quad \left(\varepsilon \left(\frac{1}{2} e^{-x} p^2 Q^* - e^{-x} pR^* + e^{-x} S^* \right) \right) e^{ib^*b} e^{\frac{b}{4}} db.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \sup_{x,p} \int_{\mathbb{R}^2} |F_{b^*}((t,u), (x,p))| dt du \\
& \leq \sup_{x,p} 1_{\{e^{-x} > \delta^6\}} \int_{\mathbb{R}} \frac{1}{|ib^* + \frac{1}{4}|} |(e^{-b/2} e^x (e^{-x} p) \partial_2 + \partial_B) \widehat{F^h}(E(t-x, u - e^{-\frac{b}{2}} p) \exp bB) \\
& \quad \left(\varepsilon \left(\frac{1}{2} e^{-x} p^2 Q^* - e^{-x} p R^* + e^{-x} S^* \right) \right) | e^{\frac{b}{4}} db dt du \\
& < \frac{C}{|b^*|},
\end{aligned}$$

for $|b^*| \geq 1$ and for some constant $C > 0$. Furthermore,

$$\begin{aligned}
& \sup_{t,u} \int_{\mathbb{R}^2} |F_{b^*}((u,v), (x,p))| dx dp \\
& \leq \int_{\mathbb{R}} \left| \frac{1}{ib^* + \frac{1}{4}} \right| 1_{\{e^{-x} > \delta^6\}}(x) |(e^{-b/2} e^x (e^{-x} p) \partial_2 + \partial_B) \widehat{F^h}(E(t-x, u - e^{-\frac{b}{2}} p) \exp bB) \\
& \quad \left(\varepsilon \left(\frac{1}{2} e^{-x} p^2 Q^* - e^{-x} p R^* + e^{-x} S^* \right) \right) | e^{\frac{b}{4}} db dx dp \\
& < \frac{C}{|b^*|},
\end{aligned}$$

again because the function

$$(t, u, x, p, b) \mapsto 1_{\{e^{-x} > \delta^6\}} \widehat{F^h}(E(t-x, u - e^{-\frac{b}{2}} p) \exp(bB)) \left(\varepsilon \left(\frac{1}{2} p^2 Q^* - p R^* + S^* \right) e^{-x} \right) e^{\frac{b}{4}}$$

has compact support and is C^∞ in the variables t, u, p, b . Hence, by the Young's inequality, for $F \in L_c^1$, there exists a constant $C > 0$ such that $\|\sigma_{b^* B^* + \varepsilon S^*}(F)\|_{\text{op}} \leq \frac{C}{|b^*|}$ for large enough $b^* \in \mathbb{R}^*$. \square

Proposition 6.19. *For any $F \in L_c^1$, we have that*

$$\lim_{\delta \rightarrow 0} \|\sigma_{\varepsilon S^*}(F)(1 - M_{\delta,1}) - \sigma_{\varepsilon, \delta, D}(F)\|_{\text{op}} = 0.$$

Proof. Let $F \in L_c^1$. Then there exists $\rho > 0$ such that

$$\hat{F}^h \left(s - x, t - b, e^{\frac{-b}{2}}(v - p), \left(\frac{e^{-x-b} p^2}{2} \right) \varepsilon Q^* + (e^{-\frac{2x+b}{2}} p) \varepsilon R^* + e^{-x} \varepsilon S^* \right) = 0$$

if $e^{-x-b} p^2 > \rho$. For δ small enough, from Proposition 6.14 we have that $P_{\delta, D, 3, k} \circ \sigma_{\varepsilon S^*}(F) \circ M_{\delta, D, 3, k} = \sigma_{\varepsilon S^*}(F) \circ M_{\delta, D, 3, k}$, $k \in \mathbb{Z}$. Therefore, we must show that for $k \in \mathbb{Z}$,

$$\|P_{\delta, D, 3, k} \circ \sigma_{\varepsilon S^*}(F) \circ M_{\delta, D, 3, k} - P_{\delta, D, 3, k} \circ \sigma_{k_3^2 \delta^4 D^2 e^{r_\delta(k_1+k_2)} \varepsilon Q^*}(F) \circ M_{\delta, D, 3, k}\|_{\text{op}} \leq C\delta,$$

for some constant $C > 0$ which is independent of δ and k .

The kernel function $F_{\delta,D,3,k}$ of the operator $P_{\delta,D,3,k} \circ \sigma_{\varepsilon S^*}(F) \circ M_{\delta,D,3,k} - P_{\delta,D,3,k} \circ \sigma_{(\frac{k_3^2 \delta^4 D^2 e^{r_\delta(k_1+k_2)}}{2})\varepsilon Q^*}(F) \circ M_{\delta,D,3,k}$ is given by

$$\begin{aligned} F_{\delta,D,3,k}((s, t, v), (x, b, p)) &= \left(\widehat{F}^h(s-x, t-b, e^{\frac{-b}{2}}(v-p), (\frac{e^{-x-b}p^2}{2})\varepsilon Q^* + (e^{-\frac{2x+b}{2}}p)\varepsilon R^* + e^{-x}\varepsilon S^*) \right. \\ &\quad \left. - \widehat{F}^h(s-x, t-b, e^{\frac{-b}{2}}(v-p), e^{-x-b}\frac{k_3^2 \delta^4 D^2 e^{r_\delta(k_1+k_2)}}{2}\varepsilon Q^*) \right) \\ &\quad e^{-\frac{b}{2}}1_{S_{\delta,D,3,k}}(x, b, p)1_{R_{\delta,D,3,k}}(s, t, v). \end{aligned}$$

Again this gives us the estimate

$$\begin{aligned} |F_{\delta,D,3,k}((s, t, v), (x, b, p))| &\leq \varphi(s-x, t-b, e^{\frac{-b}{2}}(v-p))e^{-\frac{b}{2}} \\ &\quad (|e^{\frac{-x-b}{2}}(p - \delta^2 k_3 D e^{r_\delta(k_1+k_2)/2})| |e^{\frac{-x-b}{2}}(p + \delta^2 k_3 D e^{r_\delta(k_1+k_2)/2})| \\ &\quad + e^{\frac{-2x-b}{2}}|p| + e^{-x}), \end{aligned}$$

where φ is as in Remark 6.1. On the sets $S_{\delta,D,3,k}$, if $\widehat{F}^h(-) \neq 0$ for δ small enough, we have the relations:

$$\begin{aligned} e^{-x} &\leq \delta^6, \\ e^{-x-b}p^2 &\leq \rho, \\ e^{1/2(-x-b)}|k_3|\delta^2 D e^{1/2r_\delta(k_1+k_2)} &\leq e^{1/2(-x-b)}(|p| + D\delta^2 e^{1/2r_\delta(k_1+k_2)}) \\ &\leq \rho^{1/2} + e^{1/2(-x-b)}D\delta^2 e^{1/2r_\delta(k_1+k_2)} \\ &< \rho^{1/2} + \delta \leq 2\rho^{1/2}, \\ e^{1/2(-x-b)}(p - k_3 D \delta^2 e^{1/2r_\delta(k_1+k_2)}) &\leq D\delta^2 e^{(-x-b)/2} e^{1/2r_\delta(k_1+k_2)} \\ &< D\delta^{3/2}. \end{aligned}$$

Therefore,

$$\begin{aligned} |F_{\delta,D,3,k}((s, t, v), (x, b, p))| &\leq \varphi(s-x, t-b, e^{\frac{-b}{2}}(v-p))e^{-\frac{b}{2}}(2\rho^{1/2}D\delta^{3/2} + \rho^{1/2}\delta^3 + \delta^6) \\ &\leq \delta\varphi(s-x, t-b, e^{\frac{-b}{2}}(v-p))e^{-\frac{b}{2}}. \end{aligned}$$

□

Therefore we can conclude as in Subsection 6.1.2.

Corollary 6.20. *Let $a \in C^*(G)$. Then*

$$\lim_{\delta \rightarrow 0} \text{dis}(\rho_{\varepsilon S^*}(\sigma_{\varepsilon S^*}(a) - \sigma_{\varepsilon, \delta, D}(a)), C_0(\mathbb{R}, \mathcal{K})) = 0.$$

Proof. Indeed, by Proposition 6.19, we know that $\lim_{\delta \rightarrow 0} \|\sigma_{\varepsilon S^*}(a) \circ (1 - M_{\delta,1}) - \sigma_{\varepsilon, \delta, D}(a)\|_{\text{op}} = 0$ for any $a \in C^*(G)$. By Proposition 6.18, we know that $\tau_{\varepsilon S^*}(a) \circ$

$\overline{M}_{\delta,1} \in C_0(\mathbb{R}, \mathcal{K})$. Hence, by Proposition 6.15 and relation (6.2.1):

$$\begin{aligned}
& \text{dis}(\rho_{\varepsilon S^*}(\sigma_{S^*}(a) - \sigma_{\varepsilon, \delta, D}(a)), C_0(\mathbb{R}, \mathcal{K})) \\
& \leq \|\rho_{\varepsilon S^*}(\sigma_{S^*}(a) - \sigma_{\varepsilon, \delta, D}(a)) - \tau_{\varepsilon S^*}(a) \circ \overline{M}_{\delta,1}\|_{\text{op}} \\
& = \|\rho_{\varepsilon S^*}(\sigma_{S^*}(a) \circ M_{\delta,1} + \sigma_{S^*}(a) \circ (1 - M_{\delta,1}) - \sigma_{\varepsilon, \delta, D}(a)) - \tau_{\varepsilon S^*}(a) \circ \overline{M}_{\delta,1}\|_{\text{op}} \\
& = \|\sigma_{\varepsilon S^*}(a) \circ (1 - M_{\delta,1}) - \sigma_{\varepsilon, \delta, D}(a)\|_{\text{op}} \\
& \rightarrow 0 \quad \text{as} \quad \delta \rightarrow 0.
\end{aligned}$$

□

6.3 The boundary condition for $\mathbb{R}_{\varepsilon R^*}$

This case is similar to the preceding one (and easier). We shall omit most of the proofs.

We take the coordinates on G coming from the basis $\{X := A + B, Y := A - B, P, Q, R, S\}$. Then

$$[X, P] = 0, [Y, P] = P, [X, Q] = Q, [Y, Q] = -Q, [X, R] = R, [Y, R] = 0.$$

Hence for $(x, y, p) := \exp(xX) \exp(yY) \exp(pP) \in G_0$, we have

$$(x, y, p) \cdot \varepsilon R|_{\mathfrak{h}}^* = -e^{-x+y} p \varepsilon Q^* + e^{-x} \varepsilon R^*.$$

The irreducible representation $\pi_{\varepsilon R^* + b^* B^*}$ is realized as $\pi_{\varepsilon R^* + b^* B^*} := \text{ind}_L^G \chi_{\varepsilon R^* + b^* B^*}$, where $L := \exp(\text{span}\{Y, \mathfrak{h}\})$. Let $\partial \mathbb{R}_{\varepsilon R^*}$ be the boundary of $\mathbb{R}_{\varepsilon R^*}$ in \mathfrak{g}^*/G . It follows from the description of the coadjoint orbits (see (4.1.4)) that

$$\partial \mathbb{R}_{\varepsilon R^*} = (\{S, R\}^\perp)/G.$$

Definition 6.21. *Let*

$$\sigma_{\varepsilon R^*} := \text{ind}_H^G \chi_{\varepsilon R^*}.$$

The kernel function \mathcal{K}_F of the operator $\sigma_{\varepsilon R^*}(F)$ is then given by

$$\mathcal{K}_F(s, t, v; x, y, p) = \widehat{F}^{\mathfrak{h}}(s - x, t - y, e^y(v - p), -e^{-x+y} p \varepsilon Q^* + e^{-x} \varepsilon R^*) e^y.$$

This representation is equivalent to the direct integral representation

$$\tau_{\varepsilon R^*} := \int_{\mathbb{R}}^{\oplus} \pi_{b^* B^* + \varepsilon R^*} db^*$$

acting on the Hilbert space

$$\mathcal{H}_{\tau_{\varepsilon R^*}} = \int_{\mathbb{R}}^{\oplus} L^2(G/L, \chi_{b^* B^* + \varepsilon R^*}) db^* \simeq \int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}^2) db^*$$

with the norm

$$\|\xi\|_2^2 = \int_{\mathbb{R}} \|\xi(b^*)\|_2^2 db^* \quad \text{for } \xi \in \mathcal{H}_{\tau_{\varepsilon R^*}}.$$

An intertwining operator $U_{\varepsilon R^*}$ for this equivalence is given by

$$U_{\varepsilon R^*}(\xi)(b^*)(g) := \int_{\mathbb{R}} \xi(g \exp(yY)) e^{-\frac{1}{2}b} e^{-2\pi i b^* y} dy, \quad \xi \in L^2(G/H, \chi_{\varepsilon R^*}), g \in G, b^* \in \mathbb{R}.$$

Similar to Subsection 6.2, we can again consider the C^* -algebra $C(\mathbb{R}, \mathcal{B})$ and see that the image of $\tau_{\varepsilon R^*}$ is contained in $C(\mathbb{R}, \mathcal{B})$. Similarly, we can also consider $C(\mathbb{R}, \mathcal{B})$ as a subalgebra of $B(\mathcal{H}_{\tau_{\varepsilon R^*}})$: for $\phi \in C(\mathbb{R}, \mathcal{B})$, let

$$\phi(\xi)(b^*) := \phi(b^*)(\xi(b^*)) \quad \text{for } \xi \in \mathcal{H}_{\tau_{\varepsilon R^*}}, b^* \in \mathbb{R}.$$

The unitary mapping $U_{\varepsilon R^*}$ induces a canonical homomorphism $\rho_{\varepsilon R^*}$ from the algebra $B(L^2(G/H, \chi_{\varepsilon R^*}))$ onto $B(\mathcal{H}_{\tau_{\varepsilon R^*}})$. This homomorphism is defined on $\sigma_{\varepsilon R^*}(a)$ by

$$\begin{aligned} \rho_{\varepsilon R^*}(\sigma_{\varepsilon R^*}(a)) &= U_{\varepsilon R^*} \circ \sigma_{\varepsilon R^*}(a) \circ U_{\varepsilon R^*}^* \\ &= \int_{\mathbb{R}}^{\oplus} \pi_{b^* B^* + \varepsilon R^*}(a) db^* \\ &= \tau_{\varepsilon R^*}(a) \quad \text{for } a \in C^*(G). \end{aligned} \tag{6.3.1}$$

Definition 6.22. 1. Let $S_{\delta,1} := \{(x, y, p); e^{-x} > \delta^6\}$.

2. Let $\delta \mapsto r_{\delta} \in \mathbb{R}^+$ be such that $\lim_{\delta \rightarrow 0} r_{\delta} = +\infty$ and $\lim_{\delta \rightarrow 0} e^{r_{\delta}} \delta^{1/2} = 0$.

3. For a constant $D > 0$ and $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$, let

$$S_{\delta,D,2,k} := \{(x, y, p) \in \mathbb{R}^3; e^{-x} \leq \delta^6, x \in I_{r_{\delta}, k_1}, y \in I_{r_{\delta}, k_2}, p \in I_{D\delta^2 e^{r_{\delta}(k_1 - k_2)}, k_3}\}.$$

Proposition 6.23. For every compact subset $K \subseteq \mathbb{R}^3$, we have that

$$KS_{\delta,D,2,k} \subset \bigcup_{\substack{j \in \mathbb{Z}^3 \\ |j_i| \leq 1, i=1,2,3}} S_{\delta, D e^{r_{\delta}(-j_1 - j_2)}, 2, k+j} =: R_{\delta,D,2,k}$$

for every $k \in \mathbb{Z}^3$ and $\delta > 0$ small enough.

Proof. Indeed, we have an $M > 0$ such that $K \subset [-M, M]^3$ and then for $r_{\delta} > M$, $(s, t, v) \in K$ and $(x, y, p) \in S_{\delta,D,2,k}$. It follows that

$$u := (s, t, v) \cdot (x, y, p) = (s + x, t + y, e^{-y}v + p),$$

$(k_1 + j_1)r_{\delta} \leq s + x < (k_1 + j_1 + 1)r_{\delta}$ and $(k_1 + j_1)r_{\delta} \leq t + y < (k_2 + j_2 + 1)r_{\delta}$ for some $k = (k_1, k_2) \in \mathbb{Z}^2$ and $j_1, j_2 \in \{-1, 0, 1\}$. It follows that

$$\begin{aligned} |e^{-y}v| &\leq |e^{-x}v| e^{x-y} \\ &< D e^{r_{\delta}(-j_1 + j_2)} \delta^2 e^{r_{\delta}(k_1 - k_2 + j_1 - j_2)}, \end{aligned}$$

since for δ small enough $M\delta^6 e^{2r_\delta} < D\delta^2$. Hence,

$$\begin{aligned} p + e^{-y}v &< (k_3 + 1)De^{r_\delta(-j_1+j_2)}\delta^2 e^{r_\delta(k_1-k_2+j_1-j_2)} + e^{-y}v \\ &< (k_3 + 2)De^{r_\delta(-j_1+j_2)}\delta^2 e^{r_\delta(k_1-k_2+j_1-j_2)} \end{aligned}$$

and also

$$\begin{aligned} p + e^{-y}v &\geq k_3De^{r_\delta(-j_1+j_2)}\delta^2 e^{r_\delta(k_1-k_2+j_1-j_2)} - e^{-y}|v| \\ &\geq r_\delta(k_3 - 1)De^{r_\delta(-j_1+j_2)}\delta^2 e^{r_\delta(k_1-k_2+j_1-j_2)}. \end{aligned}$$

Hence u is contained in the set $R_{\delta,D,2,k}$. \square

Define the corresponding multiplication operators on $L^2(\mathbb{R}^2)$ by $M_{\delta,1} := M_{S_{\delta,1}}$, $M_{\delta,D,k,2} := M_{S_{\delta,D,k,2}}$ and $P_{\delta,D,2,k} := M_{R_{\delta,D,2,k}}$ for all $k \in \mathbb{Z}^3$.

Proposition 6.24. *We have that*

$$\overline{M}_{\delta,1} \circ U_{\varepsilon R^*} = U_{\varepsilon R^*} \circ M_{\delta,1} \quad \text{for } \delta > 0,$$

where $\overline{M}_{\delta,1} := M_{\{(x,p); e^{-x} > \delta^6\}}$.

Proof. Indeed, for $\xi \in L^2(G/H, \chi_{\varepsilon R^*})$, $y \in \mathbb{R}$ and $b \in \mathbb{R}$, we have that

$$\begin{aligned} U_{\varepsilon R^*}(M_{\delta,1}(\xi))(E(x,p)) &= \int_{\mathbb{R}} M_{\delta,1}(\xi)(E(x,p) \exp(yY)) e^{-\frac{1}{2}y} e^{-2\pi i b^* y} dy \\ &= \int_{\mathbb{R}} 1_{\{e^{-x} > \delta^6\}} \xi(E(x,p) \exp(yY)) e^{-\frac{1}{2}y} e^{-2\pi i b^* y} dy \\ &= 1_{\{e^{-x} > \delta^6\}}(E(x,p)) \int_{\mathbb{R}} \xi(E(x,p) \exp(yY)) e^{-\frac{1}{2}y} e^{-2\pi i b^* y} dy \\ &= \overline{M}_{\delta,1}(U_{\varepsilon R^*}\xi)(E(x,p)). \end{aligned}$$

\square

Definition 6.25. For $D > 0$ and $F \in L_c^1$, let

$$\sigma_{\varepsilon,\delta,D,2}(F) := \sum_{k \in \mathbb{Z}^3} P_{\delta,D,2,k} \circ \sigma_{\varepsilon\delta^2 k_3 D e^{r_\delta(k_1-k_2)} Q^*}(F) \circ M_{\delta,D,2,k}.$$

The kernel functions $F_{\varepsilon,\delta,D,2}$ of this operator is given by :

$$\begin{aligned} F_{\varepsilon,\delta,D,2}((s,t,v), (x,y,p)) &= \sum_{k \in \mathbb{Z}^3} F_{\varepsilon,\delta,D,2,k}((s,t,v), (x,y,p)) \\ &:= \sum_{k \in \mathbb{Z}^3} \widehat{F}^h(s-x, t-y, e^{-y}(v-p), e^{-x+y}\delta^2 k_3 D e^{r_\delta(k_1-k_2)} \varepsilon Q^*) \\ &\quad e^{-y} 1_{R_{\delta,D,2,k}}(s,t,v) 1_{S_{\delta,D,2,k}}(x,y,p). \end{aligned}$$

Proposition 6.26. *For every $F \in L_c^1$ and every small enough $\delta > 0$, there exists a constant $C > 0$ such that*

$$\|\sigma_{\varepsilon,\delta,D,2,k}(F)\|_{\text{op}} \leq C \|\sigma_{\varepsilon Q^*}(F)\|_{\text{op}}.$$

Proof. The proof is similar to the proof of Proposition 6.17. \square

Proposition 6.27. *Let $a \in C^*(G)$. Then the element $\tau_{\varepsilon R^*}(a) \circ \overline{M}_{\delta,1} := \int_{\mathbb{R}}^{\oplus} \pi_{b^* B^* + \varepsilon R^*}(a) \circ \overline{M}_{\delta,1} db^*$ is in $C_0(\mathbb{R}, \mathcal{K})$ for every $\delta > 0$.*

Proof. Let us prove that the operators $\pi_{b^* B^* + \varepsilon R^*}(a) \circ \overline{M}_{\delta,1}, \delta > 0$, are all compact. However, it is easy to see, using Remark 6.1, that for $F \in L_c^1$:

$$\begin{aligned} & \|\pi_{b^* B^* + \varepsilon R^*}(F) \circ \overline{M}_{\delta,1}\|_{H-S}^2 \\ &= \int_{\mathbb{R}^4} 1_{\{e^{-x} > \delta^6\}} \left| \int_{\mathbb{R}} e^{ib^* y} e^{-\frac{y}{2}} \widehat{F}^h(E(t-x, u - e^y p) \exp yY) (-\varepsilon p e^{-x+y} Q^* + \varepsilon e^{-x} R^*) dy \right|^2 dx dt dudp \\ &\leq \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^4} 1_{\{e^{-x} > \delta^6\}} |\widehat{F}^h(E(t-x, u - e^y p) \exp yY) (-\varepsilon p e^{-x+y} Q^* + \varepsilon e^{-x} R^*)|^2 dx dt dudp \right)^{\frac{1}{2}} e^{-y/2} dy \right)^2 \\ &= \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^4} 1_{\{e^{-x} > \delta^6\}} |e^x \widehat{F}^h(E(t, u) \exp yY) (-\varepsilon p Q^* + \varepsilon e^{-x} R^*)|^2 dx dt dudp \right)^{\frac{1}{2}} e^{-y} dy \right)^2 \\ &< \infty. \end{aligned}$$

The proof that $\lim_{b^* \rightarrow \infty} \pi_{b^* B^* + \varepsilon R^*}(a) \circ \overline{M}_{\delta,1} db^* = 0$ is similar to that of the corresponding proof of Proposition 6.18 \square

Proposition 6.28. *For any $F \in L_c^1$, there exists a constant $K = K(D, F) > 0$ such that*

$$\|\sigma_{\varepsilon R^*}(F) \circ (1 - M_{\delta,1}) - \sigma_{\varepsilon, \delta, D, 2, k}(F)\|_{\text{op}} \leq K\delta \quad \text{for } \delta > 0.$$

Proof. Let $F \in L_c^1$. From Proposition 6.23, we know that for δ small enough, $P_{\delta, D, 2, k} \circ \sigma_{\varepsilon R^*}(F) \circ M_{\delta, D, 2, k} = \sigma_{\varepsilon R^*}(F) \circ M_{\delta, D, 2, k}$ for $k \in \mathbb{Z}^3$. Hence, it suffices to show that

$$\|P_{\delta, D, 2, k} \circ (\sigma_{\varepsilon R^*}(F) \circ M_{\delta, D, 2, k} - \sigma_{\varepsilon \delta^2 k_3 D e^{r_\delta(k_1 - k_2)} Q^*}(F) \circ M_{\delta, D, 2, k})\|_{\text{op}} \leq L\delta,$$

for some constant $L > 0$ independent of δ .

The kernel function $F_{\varepsilon, \delta, D, 2, k}$ of the operator $P_{\delta, D, 2, k} \circ (\sigma_{\varepsilon R^*}(F) \circ M_{\delta, D, 2, k} - \sigma_{\varepsilon \delta^2 k_3 D e^{r_\delta(k_1 - k_2)} Q^*}(F) \circ M_{\delta, D, 2, k})$ is given by:

$$\begin{aligned} F_{\varepsilon, \delta, D, 2, k}((s, t, v), (x, y, p)) &= \left(\widehat{F}^h(s-x, t-y, e^y(v-p), -e^{-x+y} p \varepsilon Q^* + e^{-x} \varepsilon R^*) \right. \\ &\quad \left. - \widehat{F}^h(s-x, t-y, e^y(v-p), e^{-x+y} \delta^2 k_3 D e^{r_\delta(k_1 - k_2)} \varepsilon Q^*) \right) \\ &\quad 1_{S_{\delta, D, 2, k}}(x, y, p) 1_{P_{\delta, D, 2, k}}(s, t, v) e^y. \end{aligned}$$

Hence,

$$\begin{aligned} & |F_{\varepsilon, \delta, D, 2, k}((s, t, v), (x, y, p))| \\ &\leq |\widehat{F}^h(s-x, t-y, v - e^y p, -e^{-x+y} p \varepsilon Q^* + e^{-x} \varepsilon R^*) \\ &\quad - \widehat{F}^h(s-x, t-y, v - e^y p, e^{-x+y} \delta^2 k_3 D e^{r_\delta(k_1 - k_2)} \varepsilon Q^*)| \\ &\quad 1_{S_{\delta, D, 2, k}}(x, y, p) 1_{P_{\delta, D, 2, k}}(s, t, v) e^y \\ &\leq \varphi(s-x, t-y, v - e^y p) \\ &\quad (e^{-x+y} e^{r_\delta(k_1 - k_2)} D \delta^2 + e^{-x}) e^y 1_{S_{\delta, D, 2, k}}(x, y, p) 1_{P_{\delta, D, 2, k}}(s, t, v) \\ &\leq \delta \varphi(s-x, t-y, e^y(v-p)) e^y 1_{S_{\delta, D, 2, k}}(x, y, p) 1_{P_{\delta, D, 2, k}}(s, t, v), \end{aligned}$$

for δ small enough, where φ is as in Remark 6.1. We conclude as in the preceding cases. \square

Corollary 6.29. *Let $a \in C^*(G)$. Then*

$$\lim_{\delta \rightarrow 0} \text{dis}(\rho_{\varepsilon R^*}(\sigma_{\varepsilon R^*}(a) - \sigma_{\varepsilon, \delta, D, 2}(a)), C_0(\mathbb{R}, \mathcal{K})) = 0.$$

Proof. The proof is similar to that of Corollary 6.20. \square

6.4 The boundary condition for $\Omega_{\varepsilon P^* + \nu Q^*}$, $\varepsilon = \pm 1, \nu = \pm 1$

The boundary of the orbit $\Omega_{\varepsilon P^* + \nu Q^*}$ is the union of the orbits $x^* X^* + \nu Q^*, y^* Y^* + \varepsilon P^*$ for all $y^*, x^* \in \mathbb{R}$, and of the set of characters (see (4.1.6)). We take the coordinates on G coming from the basis $\{Y := 2A, X := A + B, P, Q, R, S\}$. This gives us the bracket relations:

$$[X, P] = 0, [X, Q] = Q, [Y, P] = P, [Y, Q] = 0.$$

The irreducible representation $\pi_{\varepsilon P^* + \nu Q^*}$ is realized as $\pi_{\varepsilon P^* + \nu Q^*} := \text{ind}_L^G \chi_{\varepsilon P^* + \nu Q^*}$, where $L := \exp(\mathfrak{l}) := \exp(\text{span}\{P, \mathfrak{h}\})$.

Definition 6.30. 1. Let $\mathfrak{m} := \text{span}\{Y, \mathfrak{l}\}$ and $M := \exp(\mathfrak{m})$ and let

$$\sigma_{\nu Q^*} := \text{ind}_L^G \chi_{\nu Q^*} \simeq \tau_{\nu Q^*} := \int_{\mathbb{R}}^{\oplus} \pi_{x^* X^* + \nu Q^*} dx^*.$$

An intertwining operator $U_{\nu Q^*}$ for this equivalence is given by

$$(U_{\nu Q^*}(\xi)(x^*))(g) := \int_{\mathbb{R}} \xi(g \exp(xY)) e^{-2\pi i x^* x} dx$$

for $\xi \in L^2(G/L, \chi_{\nu Q^*})$, $g \in G$ and $x^* \in \mathbb{R}$. The unitary mapping $U_{\nu Q^*}$ induces an isomorphism $\rho_{\nu Q^*}$ from the algebra $B(L^2(G/L, \chi_{\nu Q^*}))$ onto $B(\int_{\mathbb{R}}^{\oplus} B(L^2(G/M, \chi_{x^* X^* + \nu Q^*})) dx^*)$. This homomorphism is defined by

$$\begin{aligned} \rho_{\nu Q^*}(\sigma_{\nu Q^*}(a)) &= U_{\nu Q^*} \circ \sigma_{\nu Q^*}(a) \circ U_{\nu Q^*}^* \\ &= \int_{\mathbb{R}}^{\oplus} \pi_{x^* X^* + \nu Q^*}(a) dx^* \text{ for } a \in C^*(G). \end{aligned}$$

2. Let $\mathfrak{k} := \text{span}\{X, \mathfrak{l}\}$ and $K := \exp(\mathfrak{k})$ and let

$$\sigma_{\varepsilon P^*} := \text{ind}_L^G \chi_{\varepsilon P^*} \simeq \tau_{\varepsilon P^*} := \int_{\mathbb{R}}^{\oplus} \pi_{y^* Y^* + \varepsilon P^*} dy^*.$$

An intertwining operator $U_{\varepsilon P^*}$ for this equivalence is given by

$$(U_{\varepsilon P^*}(\xi)(y^*))(g) := \int_{\mathbb{R}} \xi(g \exp(yX)) e^{-2\pi i y^* y} dy$$

for $\xi \in L^2(G/L, \chi_{\varepsilon P^*})$, $g \in G$ and $x^* \in \mathbb{R}$. The unitary mapping $U_{\varepsilon P^*}$ induces an isomorphism $\rho_{\varepsilon P^*}$ from the algebra $B(L^2(G/L, \chi_{\varepsilon P^*}))$ onto $B(\int_{\mathbb{R}}^{\oplus} B(L^2(G/K, \chi_{y^* Y^* + \varepsilon P^*})) dy^*)$. This homomorphism is defined by

$$\begin{aligned} \rho_{\varepsilon P^*}(\sigma_{\varepsilon P^*}(a)) &= U_{\varepsilon P^*} \circ \sigma_{\varepsilon P^*}(a) \circ U_{\varepsilon P^*}^* \\ &= \int_{\mathbb{R}}^{\oplus} \pi_{y^* Y^* + \varepsilon P^*}(a) dy^* \text{ for } a \in C^*(G). \end{aligned}$$

3. Let $\sigma_0 := \text{ind}_L^G \chi_0$.

Definition 6.31. 1. Let $S_{\delta,1} := \{(x, y); e^{-x} > \delta^6, e^{-y} > \delta^6\}$,

2. $S_{\delta,2} := \{(x, y); e^{-x} > \delta^6, e^{-y} \leq \delta^6\}$,

3. $S_{\delta,3} := \{(x, y); e^{-x} \leq \delta^6, e^{-y} > \delta^6\}$,

4. $S_{\delta,4} := \{(x, y); e^{-x} \leq \delta^6, e^{-y} \leq \delta^6\}$,

and as usual we denote by $M_{\delta,i}$ the multiplication operator on $L^2(\mathbb{R}^2)$ with the function $1_{S_{\delta,i}}$ for $1 \leq i \leq 4$.

Proposition 6.32. For every $a \in C^*(G)$, the operator $\pi_{\varepsilon P^* + \nu Q^*}(a) \circ M_{\delta,1}$ is compact.

Proof. Let $F \in L_c^1 := \{F \in L^1(G), \widehat{F}^L \in C_c(G/L, \mathfrak{l}^*)\}$. The kernel function $F_{\varepsilon, \nu}$ of the operator $\pi_{\varepsilon P^* + \nu Q^*}(F) \circ M_{\delta,1}$ is given by

$$F_{\varepsilon, \nu}((s, t), (x, y)) = (\widehat{F}^L(s - x, t - y), e^{-x} P^* + e^{-y} Q^*) 1_{S_{\delta,1}}(x, y).$$

Since $F \in L_c^1$ is of compact support in all variables and therefore $\pi_{\varepsilon P^* + \nu Q^*}(F) \circ M_{\delta,1}$ is Hilbert-Schmidt. Since L_c^1 is dense in $C^*(G)$, we have that $\pi_{\varepsilon P^* + \nu Q^*}(a) \circ M_{\delta,1}$ is compact for every $a \in C^*(G)$. \square

Definition 6.33. For $F \in L_c^1$ and $\delta > 0$, let

$$\sigma_{\varepsilon, \nu, \delta}(F) := \sigma_{\nu Q^*}(F) \circ M_{\delta,2} + \sigma_{\varepsilon P^*}(F) \circ M_{\delta,3} + \sigma_0(F) \circ M_{\delta,4} \in \mathcal{B}(L^2(\mathbb{R}^2)).$$

The kernel function $F_{\varepsilon, \nu, \delta}$ of this operator is given by

$$\begin{aligned} F_{\varepsilon, \nu, \delta}((s, t), (x, y)) &= \widehat{F}^L((s - x, t - y), e^{-x} P^*) 1_{S_{\delta,2}}(x, y) \\ &\quad + \widehat{F}^L((s - x, t - y), e^{-y} Q^*) 1_{S_{\delta,3}}(x, y) + \widehat{F}^L((s - x, t - y), 0) 1_{S_{\delta,4}}(x, y). \end{aligned}$$

Similar to previous cases, we have the following propositions and corollary.

Proposition 6.34. For every $F \in L_c^1$ and every $\delta > 0$,

$$\|\sigma_{\varepsilon, \nu, \delta}(F)\|_{\text{op}} \leq \|\sigma_{\varepsilon P^*}(F)\|_{\text{op}} + \|\sigma_{\nu Q^*}(F)\|_{\text{op}} + \|\sigma_0(F)\|_{\text{op}}.$$

Proposition 6.35. For any $F \in L_c^1$, there exists a constant $K = K(F) > 0$ such that

$$\|\pi_{\varepsilon P^* + \nu Q^*}(F) \circ (1 - M_{\delta,1}) - \sigma_{\varepsilon, \nu, \delta}(F)\|_{\text{op}} \leq K\delta \text{ for } \delta > 0.$$

Proof. Let $F \in L_c^1$. The kernel function $F_{\varepsilon, \nu, \delta, 0}$ of the operator $\pi_{\varepsilon P^* + \nu Q^*}(F) \circ (1 - M_{\delta, 1}) - \sigma_{\varepsilon, \nu, \delta}(F)$ is given by:

$$\begin{aligned} & F_{\varepsilon, \nu, \delta, 0}((s, t), (x, y)) \\ = & (\widehat{F}^L((s-x, t-y), e^{-x}P^* + e^{-y}Q^*) - \widehat{F}^L((s-x, t-y), e^{-x}P^*))1_{S_{\delta, 2}}(x, y) \\ & + (\widehat{F}^L((s-x, t-y), e^{-x}P^* + e^{-y}Q^*) - \widehat{F}^L((s-x, t-y), e^{-y}Q^*))1_{S_{\delta, 3}}(x, y) \\ & + (\widehat{F}^L((s-x, t-y), e^{-x}P^* + e^{-y}Q^*) - \widehat{F}^L((s-x, t-y), 0))1_{S_{\delta, 4}}(x, y). \end{aligned}$$

Since $F \in L_c^1$, there exists a function $\varphi \in C_c(\mathbb{R}^2)$ such that

$$|\widehat{F}((s, t), q)| \leq |\varphi(s, t)| \|q\| \quad \text{for } q \in \mathfrak{l}^*, (s, t) \in \mathbb{R}^2.$$

Therefore,

$$|F_{\varepsilon, \nu, \delta, 0}((s, t), (x, y))| \leq 3|\varphi(s-x, t-y)|\delta^6.$$

□

Corollary 6.36. *Let $a \in C^*(G)$. Then*

$$\lim_{\delta \rightarrow 0} \text{dis}(\pi_{\varepsilon P^* + \nu Q^*}(a) - \sigma_{\varepsilon, \nu, \delta}(a), \mathcal{K}) = 0.$$

6.5 The boundary conditions for $\mathbb{R}_{\varepsilon P^*}$ and $\mathbb{R}_{\nu Q^*}$

We consider only the case $\Omega_{a^*A^* + \nu Q^*}$. The other cases are similar.

We take the coordinates on G coming from the basis $\{A, B, P, Q, R, S\}$. Let $L := \exp(\mathfrak{l})$ and $\mathfrak{l} := \text{span}\{P, Q, R, S\}$. Let $\sigma_0 := \text{ind}_L^G 1$ be the left regular representation of G on $L^2(G/L)$. This representation is equivalent to $\tau_0 := \oint_{\mathbb{R}} \pi_{a^*A^* + b^*B^*} da^* db^*$ acting on $L^2(\mathbb{R}^2) = \oint_{\mathbb{R}^2} \mathbb{C} da^* db^*$. Let $U_{\nu Q^*}$ be the unitary equivalence given by

$$U_{\nu Q^*}(\xi)(a^*, b^*) := \int_{\mathbb{R}^2} \xi(\exp(aA) \exp(bB)) e^{-2\pi i(aa^* + bb^*)} da db, \quad a^*, b^* \in \mathbb{R}.$$

We see that the algebra $B(L^2(G/L, \chi_{\nu Q^*}))$ is mapped into $C(\mathbb{R}^2) \subset \mathcal{B}(L^2(\mathbb{R}^2))$ by the mapping

$$\rho_{\nu Q^*}(\xi) := U_{\nu Q^*} \circ \xi \circ U_{\nu Q^*}^*.$$

Definition 6.37. For $\delta > 0$, let

1. $S_{\delta, 1} := \{(a, b) \in \mathbb{R}^2; e^{-b} > \delta\},$
2. $S_{\delta, 2} := \{(a, b) \in \mathbb{R}^2; e^{-b} \leq \delta\},$

and $M_{\delta, i}$ denotes the multiplication operator on $L^2(\mathbb{R}^2)$ with the function $1_{S_{\delta, i}}$ for $i = 1, 2$. For each $F \in C^*(G)$, we define the linear operator $\sigma_{\delta, 2}(F)$ on $L^2(\mathbb{R}^2)$ by

$$\sigma_{\delta, 2}(F) := \sigma_0(F) \circ M_{\delta, 2}.$$

Proposition 6.38. *Let $a \in C^*(G)$. Then*

$$\lim_{\delta \rightarrow 0} \text{dis}((\sigma_{\nu Q^*}(a) \circ (1 - M_{\delta,1}) - \sigma_{\delta,2}(a)), C_0(\mathbb{R}, \mathcal{K})) = 0.$$

The proof is similar to that of the corresponding proof in Section 6.3.

7 The C*-algebra of G

We recall that the space \widehat{G} has been identified in Definition 5.6 with the disjoint union of the subsets $\Gamma_j, j = 0, \dots, 6$, of \mathfrak{g}^* . We let

$$l^\infty(\widehat{G}) := \{(\phi(\gamma))_{\gamma \in \widehat{G}}; \phi(\gamma) \in B(\mathcal{H}_{\pi_\gamma}), \|\phi\| := \sup_{\gamma \in \widehat{G}} \|\phi(\gamma)\|_{\text{op}} < \infty\}.$$

Definition 7.1. For a subset Γ of \widehat{G} and $\phi \in l^\infty(\widehat{G})$, let

$$\phi|_\Gamma := (\phi(\gamma))_{\gamma \in \Gamma} \in l^\infty(\Gamma).$$

Definition 7.2. For $F \in C^*(G)$ and $\gamma \in \widehat{G}$, we define the Fourier transform $\mathcal{F}_G : C^*(G) \rightarrow l^\infty(\widehat{G})$ by

$$\mathcal{F}_G(F)(\gamma) := \pi_\gamma(F) \in B(\mathcal{H}_{\pi_\gamma}).$$

Let

$$E^*(G) := \{\phi \in l^\infty(\widehat{G}); \text{ the mappings } \gamma \mapsto \phi(\gamma) \text{ are norm continuous and} \\ \text{vanish at infinity on the sets } \Gamma_i, i = 1, 2, 4, 5\}.$$

Then the subset $E^*(G)$ is a C*-subalgebra of $l^\infty(\widehat{G})$ containing $\mathcal{F}_G(C^*(G))$ by Section 5. We can define the following representations of $E^*(G)$ for $\varepsilon = \pm 1$:

1. $\tau_{\varepsilon S^*}(\phi) := \int_{\mathbb{R}}^{\oplus} \phi(b^* B^* + \varepsilon S^*) db^*$ on the Hilbert space $\int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}^2) db^*$. We have seen in Definition 6.11 that the restriction of this representation to $\mathcal{F}_G(C^*(G))$ is equivalent to the representations $\sigma_{\varepsilon s^* S^*} = \text{ind}_H^G \chi_{\varepsilon s^* S^*}$ for $s^* \in \mathbb{R}^+$. We use now these equivalences to extend the representations $\sigma_{\varepsilon s^* S^*}$ to $E^*(G)$.
2. $\tau_{\varepsilon R^*}(\phi) := \int_{\mathbb{R}}^{\oplus} \phi(b^* B^* + \varepsilon R^*) db^*$ on the Hilbert space $\int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}^2) db^*$. This gives us representations $\sigma_{\varepsilon R^*}$ of $E^*(G)$ which coincides with $\text{ind}_H^G \chi_{\varepsilon R^*}$ on $C^*(G)$.
3. $\tau_{\varepsilon P^*}(\phi) := \int_{\mathbb{R}}^{\oplus} \phi(x^* (\frac{A^* + B^*}{2}) + \varepsilon P^*) dx^*$ on the Hilbert space $\int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}) dx^*$. This gives us representations $\sigma_{\varepsilon P^*}$ of $E^*(G)$ which coincides with $\text{ind}_{[G,G]}^G \chi_{\varepsilon P^*}$ on $C^*(G)$.
4. $\tau_{\varepsilon Q^*}(\phi) := \int_{\mathbb{R}}^{\oplus} \phi(a^* A^* + \varepsilon Q^*) da^*$ on the Hilbert space $\int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}) da^*$. This gives us representations $\sigma_{\varepsilon Q^*}$ of $E^*(G)$, which coincides with $\text{ind}_{[G,G]}^G \chi_{\varepsilon Q^*}$ on $C^*(G)$.

This allows us to define the following bounded linear mappings of $E^*(G)$, where $\varepsilon = \pm 1, \nu = \pm 1$ and $\delta > 0$:

1. (with the notations in Section 6.1.2)

$$\sigma_{\varepsilon S^* + \varepsilon Q^*, \delta}(\phi) := \sigma_{\varepsilon S^*}(\phi) \circ M_{\tilde{\varphi}_1} \circ M_{\delta,1} + \sigma_{\varepsilon S^*}(\phi) \circ M_{\delta,2} + \sigma_{\delta,3}(\phi),$$

$$\text{where } \sigma_{\delta,3}(\phi) := \sum_{k \in \mathbb{Z}} N_{\delta,k,3} \circ \sigma_{\varepsilon(1 + \frac{\delta^4 k^2}{2}) Q^*}(\phi) \circ M_{\delta,3,k};$$

2. (with the notations in Section 6.1.1)

$$\sigma_{\varepsilon S^* - \varepsilon Q^*, \delta}(\phi) := \sum_{i=1}^2 \sigma_{\varepsilon S^*}(\phi) \circ M_{\tilde{\varphi}_i} \circ M_{\delta,i} + \sigma_{\pm \sqrt{2} R^* - 2 Q^*}(\phi) \circ M_{\delta,4,\pm} + \sigma_{\delta,3}(\phi),$$

$$\text{where } \sigma_{\delta,3}(\phi) := \sum_{k \in \mathbb{Z}} N_{\delta,3,k} \circ \sigma_{\varepsilon(-1 + \frac{\delta^4 k^2}{2}) Q^*}(\phi) \circ (M_{\delta,3,k,+} + M_{\delta,3,k,-});$$

3. (with the notations in Section 6.2)

$$\sigma_{\varepsilon S^*, \delta, D}(\phi) := \sum_{k \in \mathbb{Z}^3} P_{\delta,D,3,k} \circ \sigma_{k_3^2 \delta^3 D^2 e^{r_\delta(k_1+k_2)} \varepsilon Q^*}(\phi) \circ M_{\delta,D,3,k};$$

4. (with the notations in Section 6.3)

$$\sigma_{\varepsilon R^*, \delta, D}(\phi) := \sum_{k \in \mathbb{Z}^3} P_{\delta,D,2,k} \circ \sigma_{k_3 \delta^2 D e^{r_\delta(k_1-k_2)} \varepsilon Q^*}(\phi) \circ M_{\delta,D,2,k};$$

5. (with the notations in Section 6.4)

$$\sigma_{\varepsilon P^*, \nu Q^*, \delta}(\phi) := \sigma_{\nu Q^*}(\phi) \circ M_{\delta,2} + \sigma_{\varepsilon P^*}(\phi) \circ M_{\delta,3} + \sigma_0(\phi) \circ M_{\delta,4};$$

6. similarly (with the notations of Section 6.5)

$$\sigma_{\varepsilon P^*, \delta}(\phi) = \sigma_{\varepsilon P^*} \circ M_{\delta,2},$$

and the corresponding linear mapping

$$\sigma_{\nu Q^*, \delta}(\phi) = \sigma_{\nu Q^*} \circ M_{\delta,2}.$$

We define now the C^* -subalgebra $D^*(G)$ of $l^\infty(\widehat{G})$. It will follow from Theorem 2.6 that this subalgebra is the Fourier transform of $C^*(G)$.

Definition 7.3. Let $D^*(G)$ be the subset of $l^\infty(\widehat{G})$ consisting of all operator fields ϕ contained in $E^*(G)$ such that for $\varepsilon, \omega = \pm 1$,

- 1.

$$\lim_{\delta \rightarrow 0} \text{dis}(\phi(\varepsilon(S^* + Q^*)) - \sigma_{\varepsilon S^* + \varepsilon Q^*, \delta}(\phi), \mathcal{K}(L^2(\mathbb{R}^3))) = 0;$$

2.

$$\lim_{\delta \rightarrow 0} \text{dis}((\phi(\varepsilon(S^* - Q^*)) - \sigma_{\varepsilon S^* - \varepsilon Q^*, \delta}(\phi)), \mathcal{K}(L^2(\mathbb{R}^3))) = 0;$$

3.

$$\lim_{\delta \rightarrow 0} \text{dis}(\rho_{\varepsilon S^*}(\sigma_{\varepsilon S^*}(\phi) - \sigma_{\varepsilon S^*, \delta, D}(\phi)), C_0(\mathbb{R}, \mathcal{K})) = 0,$$

this is,

$$\lim_{\delta \rightarrow 0} \text{dis}(\tau_{\varepsilon S^*}(\phi) - \rho_{\varepsilon S^*}(\sigma_{\varepsilon S^*, \delta, D}(\phi)), C_0(\mathbb{R}, \mathcal{K})) = 0;$$

4.

$$\lim_{\delta \rightarrow 0} \text{dis}(\rho_{\varepsilon R^*}(\sigma_{\varepsilon R^*}(\phi) - \sigma_{\varepsilon R^*, \delta, D}(\phi)), C_0(\mathbb{R}, \mathcal{K})) = 0,$$

this is,

$$\lim_{\delta \rightarrow 0} \text{dis}(\tau_{\varepsilon R^*}(\phi) - \rho_{\varepsilon R^*}(\sigma_{\varepsilon R^*, \delta, D}(\phi)), C_0(\mathbb{R}, \mathcal{K})) = 0;$$

5.

$$\lim_{\delta \rightarrow 0} \text{dis}((\phi(\varepsilon P^* + \nu Q^*) - \sigma_{\varepsilon P^* + \nu Q^*, \delta}(\phi)), \mathcal{K}(L^2(\mathbb{R}^3))) = 0;$$

6.

$$\lim_{\delta \rightarrow 0} \text{dis}(\rho_{\varepsilon P^*}(\sigma_{\varepsilon P^*}(\phi) - \sigma_{\varepsilon P^*, \delta}(\phi)), C_0(\mathbb{R}, \mathcal{K})) = 0,$$

this is,

$$\lim_{\delta \rightarrow 0} \text{dis}(\tau_{\varepsilon P^*}(\phi) - \rho_{\varepsilon P^*}(\sigma_{\varepsilon P^*, \delta}(\phi)), C_0(\mathbb{R}, \mathcal{K})) = 0;$$

7.

$$\lim_{\delta \rightarrow 0} \text{dis}(\rho_{\nu Q^*}(\sigma_{\nu Q^*}(\phi) - \sigma_{\nu Q^*, \delta}(\phi)), C_0(\mathbb{R}, \mathcal{K})) = 0,$$

this is,

$$\lim_{\delta \rightarrow 0} \text{dis}(\tau_{\nu Q^*}(\phi) - \rho_{\nu Q^*}(\sigma_{\nu Q^*, \delta}(\phi)), C_0(\mathbb{R}, \mathcal{K})) = 0;$$

8. the same conditions hold for the adjoint field ϕ^* .

We have seen in the Sections 5 and 6 that we can use Theorem 2.6 to show the following result.

Theorem 7.4. *The C^* -algebra of the group G_6 is an almost $C_0(\mathcal{K})$ - C^* -algebra. In particular, the Fourier transform maps $C^*(G_6)$ onto the subalgebra $D^*(G_6) \subset l^\infty(\widehat{G_6})$.*

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